The Minimum Resistance Problem

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The Minimum Resistance Problem

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The Minimum Resistance Problem

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This thesis explores the Minimum Resistance Problem. The Minimum Resistance Problem seeks the three-dimensional body that gives the least resistance when subjected to fluid flow. The Minimum Resistance Problem was first posed by Newton and is arguably the oldest problem in the Calculus of Variations. Despite its age, however, the Minimum Resistance Problem is an active area of research. First, we will motivate and derive the Minimum Resistance Problem from elementary considerations. Then we will study several variants of the Minimum Resistance Problem as well as their solutions or near solutions. We will give special attention to the “radial” and ”Single Impact Condition” cases. In the radial case, the admissible bodies are both rotationally symmetric and convex. In the Single Impact Condition case, the admissible bodies satisfy a general condition that ensures fluid particles do not make multiple collisions.

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Chapter 1

The Minimum Resistance Problem

In his seminal book, *Philosophiæ Naturalis Principia Mathematica*, Sir Isaac Newton wrote:

> *If in a rare medium, consisting of equal particles freely disposed at equal distances from each other, a globe and a cylinder described on equal diameter move with equal velocities in the direction of the axis of the cylinder, (then) the resistance of the globe will be half as great as that of the cylinder... I reckon that this proposition will not be without application in the building of ships*

Newton's observation that two bodies with the same maximum cross sectional area have different resistances made him wonder which body has the lowest resistance. This, in essence, is the Minimum Resistance Problem:

> Consider the set of all three-dimensional bodies with a horizontal base $\Omega \subset \mathbb{R}^2$ in the xy plane. Suppose that a fluid flows in the $-z$ direction. Which body has minimum resistance to the fluid flow?

The Minimum Resistance Problem is arguably the oldest problem in the Calculus of Variations. The solution (if one exists) clearly depends on any constraints we put on the set of admissible bodies, the physical properties of the fluid flow, and our definition of resistance.

As such, there are countless variants of Minimum Resistance Problem. In this thesis, we will explore a few of these variants, as well as their solutions or approximate solutions. In chapter 1, we will formulate a Minimum Resistance Problem in terms of a functional, and explore a sequence of progressively more constrained variants of the Minimum Resistance Problem. In chapter 2, we will consider the Minimum Resistance Problem when the admissible bodies are convex and rotationally symmetric. This so called "radial case" was first studied by Newton. In chapter 3, we will consider the Minimum Resistance Problem when the admissible bodies satisfy the "Single Impact Condition" (which we will define in section 1.6). Plakhov, [Pla16], recently solved this variant.

1.1 Fluid Models

In this section, we will consider two fluid models and give an explicit definition of resistance. The first fluid model is due to Newton. As stated in [And09], Newton used the following basic fluid model:

- The fluid consists of a uniform stream of non-interacting particles. Each particle has the same initial velocity $\overrightarrow{v_i}$. [*]

- If a fluid particle strikes the surface of the body, then it transfers all of its normal momentum (momentum in the direction of the normal vector of the body's surface at the point of impact) to the body but preserves its tangential momentum (momentum in the direction parallel to the body's surface at the point of impact).

[*]By uniform, we mean that the fluid particles are uniformly distributed throughout space. Therefore, the flux of particles through a surface is independent of the position of that surface.

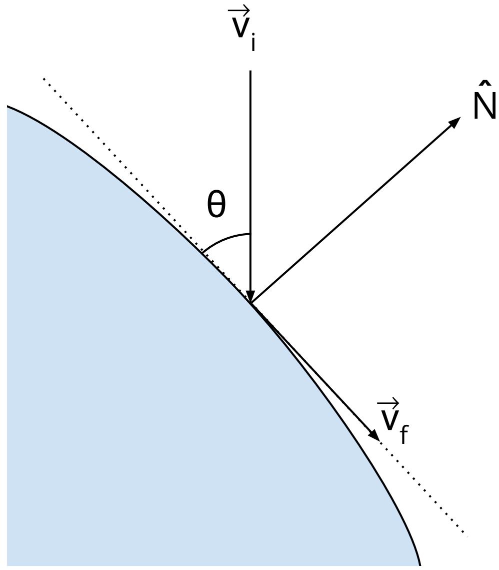

Figure 1.1: Particle impact under Newton's fluid model

Thus, after striking the body's surface, the particle will move parallel to the surface at the point of impact. Figure 1.1 depicts a particle collision under Newton's fluid model.

In general, Newton's fluid model is a crude approximation to actual physics. However, the model is a reasonable approximation for bodies that move at hypersonic (high Mach number) speeds in an ideal gas and for bodies moving at low speed in a rarefied gas. The interested reader can find a detailed treatment of Newton's fluid model and its applicability to aerodynamics in section 14.3 of [And09].

Many papers on the Minimum Resistance Problem, including [But09], [CL01b], and [Pla15], use an "elastic fluid model", which assumes the following:

- The fluid consists of a uniform stream of non-interacting particles. Each particle has the same initial velocity, $\overrightarrow{v_i}$.

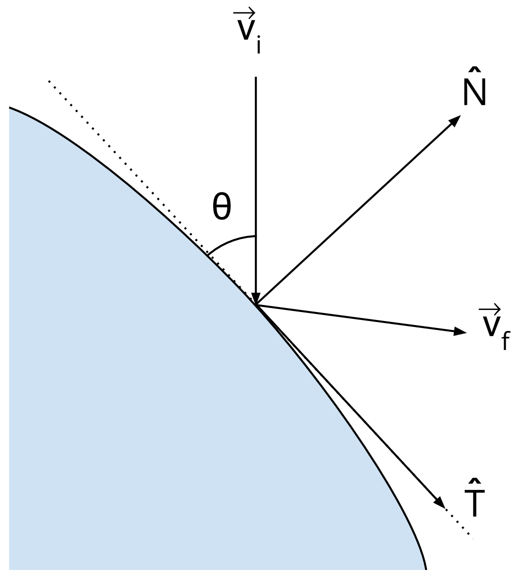

Figure 1.2: Particle impact under the elastic fluid model

- Fluid particles collide elastically with the surface such that their final speed is equal to their initial speed.
- There is no friction between the fluid and the body's surface.

Figure 1.2 depicts a particle collision under the elastic fluid model. Interestingly, both this model and Newton's lead to Newton's sine-squared pressure law, and to the same functional. In fact, the derivation using Newton's fluid model is almost identical to the one that we give in the next two sections. The key difference between the two models is how the fluid particles move after colliding with the surface. In Newton's model, particles slide along the surface after colliding with it. In the elastic fluid model, fluid particles bounce off of the surface.

This distinction will be important when we discuss the so called Single Impact Condition at the end of this chapter. The elastic fluid model engenders a general condition for the Single Impact Assumption, which will become paramount in chapter 3.

In this thesis, we will use the elastic fluid model. With this established, we can define the resistance due to a particle collision:

> **Definition 1.1**: The resistance due to a particle collision is the magnitude of the component of momentum in the direction of $\overrightarrow{v_i}$ that is transferred to the body by collision.

With this definition, and the elastic fluid model, we are ready to formulate the Minimum Resistance Problem in terms of a functional.

1.2 Deriving the Functional F

1.2.1 Newton's Sine-Squared Pressure Law

Suppose that a fluid particle strikes the body's surface at $p_0 = (x_0, y_0, z_0)$. To aid in the following discussion, let

- $\hat{N}$ denote the unit normal vector of the surface at p_0
- P denote the plane defined by $\overrightarrow{v_i}$ and $\hat{N}$ †
- $\hat{T}$ denote the unit tangent vector to the body's surface (restricted to points within P) at p_0
- θ denote the angle between $\overrightarrow{v_i}$ and $\hat{T}$

† P is not well defined if $\overrightarrow{v_i}$ and $\hat{N}$ are parallel. If this is the case, let P be any plane that contains $\hat{N}$

$\hat{N}$, $\hat{T}$, and θ, are depicted in in figure 1.2. From these definitions, we can see that

$$\vec{v_i} = |\vec{v_i}| \left(\cos(\theta)\hat{T} - \sin(\theta)\hat{N} \right) \tag{1.1}$$

Since there is no friction between the body's surface and the particles, the collision will not change the component of the particle's velocity in the direction of $\hat{T}$. Given this, and the fact that the particle's final speed must equal its initial speed, we can conclude that the component of the particle's final velocity in the direction of $\hat{N}$ is equal in magnitude and opposite in direction to the particle's initial velocity in this direction. Therefore,

$$\vec{v_f} = |\vec{v_i}| \left(\cos(\theta)\hat{T} + \sin(\theta)\hat{N} \right) \tag{1.2}$$

Looking back at figure 1.2, however, we can see that $-|\vec{v_i}|\sin(\theta)\hat{N}$ is the vector projection of $\vec{v_i}$ in the direction of $\hat{N}$. Comparing equations (1.1) and (1.2), we can see that v_i and v_f differ by twice this projection. Thus, we can rewrite (1.2) as

$$\vec{v_f} = \vec{v_i} - 2\left\langle \vec{v_i}, \hat{N} \right\rangle \hat{N} \tag{1.3}$$

Where $\langle a, b \rangle$ denotes the dot product of vectors a and b. If the particle has mass m, then the momentum transferred to the body by the collision is

$$\begin{aligned} \Delta(m\vec{V}) &= m(\vec{v_f} - \vec{v_i}) \\ &= 2m\left\langle \vec{v_i}, \hat{N} \right\rangle \hat{N} \end{aligned}$$

By definition, the resistance due to the particle collision is the component of $\Delta(m\overrightarrow{V})$ in the direction of $\overrightarrow{v_i}$. But this is exactly the dot product of $\Delta(m\overrightarrow{V})$ with $\overrightarrow{v_i}/|\overrightarrow{v_i}|$. Therefore, the resistance due to the particle collision is

$$\begin{aligned} \left\langle \Delta(m\overrightarrow{V}), \frac{\overrightarrow{v_i}}{|\overrightarrow{v_i}|} \right\rangle &= 2m\frac{\left\langle \overrightarrow{v_i}, \hat{N} \right\rangle^2}{|\overrightarrow{v_i}|} \\ &= 2m\sin(\theta)^2|\overrightarrow{v_i}| \end{aligned} \tag{1.4}$$

Thus, the resistance due to the particle is proportional to $\sin^2(\theta)$. This result is known as Newton's Sine-Squared pressure law. The interested reader can find a more detailed treatment of this result in [And09]. Newton's Sine-Squared Pressure Law tells us that, under the elastic fluid model, a body's resistance is entirely due to its geometry.

It is worth considering how the above analysis would have differed if we had instead used Newton's fluid model. Looking back at the model's definition, we can see that under Newton's fluid model, the final velocity is $\cos(\theta)\hat{T}$. Therefore, under Newton's fluid model, the momentum transfer from a particle collision is $\left\langle \overrightarrow{v_i}, \hat{N} \right\rangle = m\sin(\theta)\hat{N}$, which is exactly half of what it is under the elastic fluid model. This also means that the resistance due to a particle collision under Newton's fluid model is exactly half of what it is under the elastic fluid model.

1.2.2 The Functional F

Let $u : \Omega \to \mathbb{R}^{\geq 0}$ be differentiable almost everywhere. We can associate each u with the body $B_u = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \Omega, 0 \leq z \leq u(x, y)\}$. To find the optimal body B_u, we simply need to find the optimal u.

In this section, we express Newton's pressure law in terms of ∇u. To begin, suppose that a particle strikes the body's surface at $p_0 = (x_0, y_0, z_0)$. Let $\hat{N}$, $\hat{T}$, and P be as defined in the previous section. Suppose that u is differentiable at p_0. By this, we mean that there is a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{|f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - L(\Delta x, \Delta y)|}{|(\Delta x, \Delta y)|} = 0$$

This assumption guarantees that u, and therefore g, has a gradient at p_0. It also means that g has directional derivatives in all directions, and that $\nabla g(p_0)$ points in the direction of greatest increase.

Let us establish the following (x, y, z) coordinate system:

- Place the origin at (x_0, y_0)
- let the z axis point in direction of the normal vector of Ω.
- let the y axis point in the direction of the normal vector of P
- let the x axis be orthogonal to the y and z axes

This coordinate system is depicted in Figure 1.3.

In this coordinate system, the x and z axes are contained in P. Further, since Ω lies in the xy plane, u is a function of x and y. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by $g(x, y, z) = -u(x, y) + z$. The graph of u is, by definition, the set of points $(x, y, z) \in \mathbb{R}^3$ with $(x, y) \in \Omega$ such that $z - u(x, y) = g(x, y, z) = 0$. Thus, the graph of u is the level set corresponding to $g(x, y, z) = 0$. By inspection, points that lie above the graph of u have a larger g value than corresponding points on the graph.

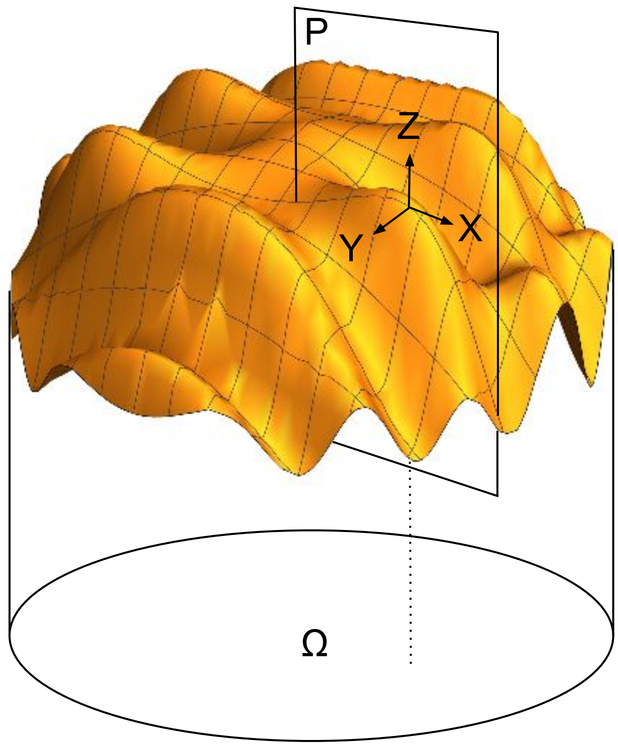

Figure 1.3: The (x, y, z) coordinate system at p_0

By definition of g,

$$\nabla g(p_0) = -\nabla u(x_0, y_0) + \hat{e_z}$$

Since u is a level set of g, we can conclude that $\nabla g(p_0)$ is orthogonal to the graph of u at p_0. Further, by inspection, the z component of $\nabla g(p_0)$ is 1, which means that $\nabla u(x_0, y_0)$ points above the graph of u. Since $\hat{N}$ points above the graph of u and is orthogonal to the graph of u at (x_0, y_0), we can conclude that $\hat{N}$ and $\nabla g(p_0)$ are parallel. Therefore,

$$\nabla \hat{g(p_0)} = \frac{\nabla g(p_0)}{|\nabla g(p_0)|} = \hat{N}$$

Substituting this into (1.4), we can conclude that the resistance due to a particle collision is

$$
\begin{aligned}
2m\frac{\left\langle \overrightarrow{v_i}, \hat{N} \right\rangle^2}{|\overrightarrow{v_i}|} &= 2m\frac{\langle \overrightarrow{v_i}, \nabla g(p_0)\rangle^2}{|\nabla g(p_0)|^2} \\
&= 2m\left(\frac{|\overrightarrow{v_i}|^2}{1+|\nabla u(x_0,y_0)|^2}\right)
\end{aligned}
\tag{1.5}
$$

Where we used $\langle \overrightarrow{v_i}, \nabla g(p_0)\rangle = -|\overrightarrow{v_i}|$, which is true because $\overrightarrow{v_i} = -|\overrightarrow{v_i}|e_z$. Therefore, we can conclude that the resistance due to the particle collision is proportional to $1/(1 + |\nabla u(x_0, y_0)|^2)$. Importantly, this result is independent of the particular collision that we considered and must, therefore, hold for every collision (assuming, of course, that u is differentiable at the point of impact). Thus, without loss of generality, we can redefine the resistance of a particle collision to be $1/(1 + |\nabla u(x_0, y_0)|^2)$.

The fact that $\hat{N} = \nabla \hat{g(p_0)}$ also tells us

$$
\begin{aligned}
\overrightarrow{v_f} &= \overrightarrow{v_i} - 2\left\langle \overrightarrow{v_i}, \hat{N} \right\rangle \hat{N} \\
&= \overrightarrow{v_i} - 2\frac{\langle \overrightarrow{v_i}, \nabla g(p_0)\rangle}{|\nabla g(p_0)|^2}\nabla g(p_0) \\
&= \overrightarrow{v_i} + 2|\overrightarrow{v_i}|\frac{\nabla g(p_0)}{|\nabla g(p_0)|^2}
\end{aligned}
$$

And thus, since $\overrightarrow{v_i}$ points in the $-z$ direction, and since $\nabla g(p_0) = -\nabla u(x_0, y_0) + \hat{e_z}$,

$$
\overrightarrow{v_f} = |\overrightarrow{v_i}|\frac{1}{1+|\nabla u(x_0,y_0)|^2}\left(-2\nabla u(x_0,y_0) + \left(1-|\nabla u(x_0,y_0)|^2\right)\hat{e_z}\right)
\tag{1.6}
$$

Thus far, we have not considered the possibility of particles colliding the surface more than once. Unfortunately, modeling multiple collisions is difficult. Therefore, we will assume that all particles impact the surface of B_u at most once. This assumption is known as the Single Impact Assumption. We will discuss this assumption in greater detail at the end of this chapter.

We want to integrate the resistance of u over Ω. To begin, let Q be a partition of Ω and consider the ith region of the partition, which we will denote by R_i. We will assume that R_i has area ΔA_i. When a particle strikes the surface of B_u in the ith region, it must do so at a point. Thus, the resistance due to this collision must lie between the supremum and infimum of of the set of resistances at points inside of the ith region. Since the fluid flow is uniform, we can conclude that the resistance due to the ith region lies between

$$\sup\left(\frac{1}{1+|\nabla u(X)|^2}, X \in R_i\right)\Delta A_i$$

and

$$\inf\left(\frac{1}{1+|\nabla u(X)|^2}, X \in R_i\right)\Delta A_i$$

Thus, the total resistance of B_u must lie between

$$\sum_i \sup\left(\frac{1}{1+|\nabla u(X)|^2}, X \in R_i\right)\Delta A_i$$

and

$$\sum_i \inf\left(\frac{1}{1+|\nabla u(X)|^2}, X \in R_i\right)\Delta A_i$$

Since we chose the partition Q arbitrarily, this must hold for every partition of Ω. In particular, this means that the resistance due to the surface is an upper bound on the set of lower sums of $1/(1+|\nabla u(X)|^2)$ and a lower bound on the set of upper sums of $1/(1+|\nabla u(X)|^2)$. Therefore, the resistance on the surface must lie between the upper and lower integrals of $1/(1+|\nabla u(X)|^2)$ on Ω. Since u is differentiable almost everywhere in Ω, $1/(1+|\nabla u(X)|^2)$ must be integrable on Ω in the Riemann sense. Therefore, we can conclude that the total resistance of B_u is proportional to,

$$F(u) = \int_\Omega \frac{1}{1+|\nabla u|^2} \tag{1.7}$$

In the next section, we will re-frame the Minimum Resistance Problem in terms of F.

1.3 The Unconstrained Minimum Resistance Problem

1.3.1 The Unconstrained Problem

Now that we've derived F, we can reformulate the Minimum Resistance Problem in terms of it. Before we do that, however, we first need to establish a set of admissible functions. At the very least, these functions need to be in the domain of F. We will also require that the upper surface of each admissible function lies above Ω. With that in mind, let

$$\mathfrak{A}_\Omega = \{u : \overline{\Omega} \to \mathbb{R}^{\geq 0} \mid u \text{ is piecewise smooth}\} \tag{1.8}$$

Importantly, if u is piecewise smooth on Ω, then it will be integrable on Ω in the Riemann sense. Thus, for each $u \in \mathfrak{A}_\Omega$, $F(u)$ is well defined. We can now state the "unconstrained" variant of the Minimum Resistance Problem.

Find $u \in \mathfrak{A}_\Omega$ which gives the lowest resistance. That is, find

$$\min\{F(v) \mid v \in \mathfrak{A}_\Omega\} \tag{1.9}$$

Unless stated otherwise, we will consider the case when Ω is the closed disk of radius $R > 0$ centered at the origin, which we will denote by $\overline{B_R(0)}$.

1.3.2 Resistance of a Globe and Cylinder

Now that we have a mathematical formulation for the Minimum Resistance Problem, we can consider Newton's original proposition: the resistance of a globe is half that of a cylinder.[‡] Let $G : \overline{B_R(0)} \to \mathbb{R}$ and $C : \overline{B_R(0)} \to \mathbb{R}$ be defined by

[‡]Newton derived this result from purely geometric considerations. The interested reader can find a detailed account of Newton's derivation in [Gol80].

$$G(X) = \sqrt{R^2 - |x|^2} \tag{1.10}$$

$$C(X) = H \tag{1.11}$$

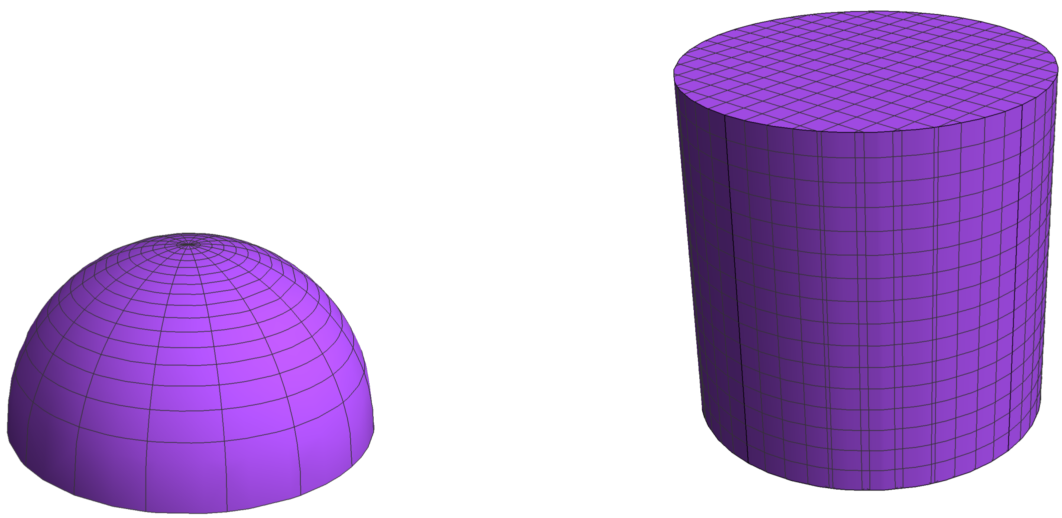

Figure 1.4: A globe, $G(X)$, and a cylinder, $C(X)$

Where $H > 0$ is the height of the cylinder C. G and C are depicted in figure 1.4. Notice that both C and G are elements of $\mathfrak{A}_\Omega$. By changing (1.7) to polar coordinates,

$$
\begin{aligned}
F(G) = \int_{B_R(0)} \frac{1}{1+|\nabla G|^2} &= \int_0^R \frac{2\pi r}{1+G'(r)^2} dr \\
&= \int_0^R \frac{2\pi r}{1+\frac{r^2}{R^2-r^2}} dr \\
&= \frac{2\pi}{R^2} \int_0^R r(R^2-r^2) dr \\
&= \frac{\pi R^2}{2}
\end{aligned}
$$

and (note: C is a constant, so $\nabla C = 0$),

$$
\begin{aligned}
F(C) &= \int_{B_R(0)} \frac{1}{1+|\nabla C|^2} \\
&= \int_{B_R(0)} 1 \\
&= \pi R^2
\end{aligned}
$$

And thus,

$$
\frac{F(G)}{F(C)} = \frac{1}{2}
$$

As predicted by Newton.

1.4 Fundamental Properties of F

The unconstrained variant of the Minimum Resistance Problem, (1.9) does not have a solution. In particular, $\inf\{F(u) \mid u \in \mathfrak{A}_{B_R(0)}\}$ is well defined, but there is no function in $\mathfrak{A}_{B_R(0)}$ whose resistance equals this infimum. We can, however, easily find approximate solutions whose resistance is arbitrarily close to the infimum. First, however, we need to establish a basic property of F.

Lemma 1.1: Let $f : \overline{B_R(0)} \to \mathbb{R}^{\geq 0}$ be integrable in Riemann sense on $B_R(0)$. If f is strictly positive on $B_R(0)$ then $\int_{B_R(0)} f > 0$.

Proof :

To begin, since f is Riemann integrable on $B_R(0)$, it must be continuous almost everywhere. Since $B_R(0)$ has a positive measure, there must be some $X \in \overline{B_R(0)}$ at which f is continuous. Since $B_R(0)$ is an open set, X must be an interior point of $B_R(0)$. Thus, there is some $\delta_1 > 0$ such that $B_{\delta_1}(X) \subset B_R(0)$. Further, since f is strictly positive on $B_R(0)$, $f(X) > 0$. Since f is continuous at x, there is some $\delta_2 > 0$ such that if $Y \in B_R(0)$ satisfies $|Y - X| < \delta_2$ then,

$$f(X) - f(Y) \leq |f(X) - f(Y)| < \frac{f(X)}{2}$$

Which implies that $f(X)/2 < f(Y)$. Now, let $\delta = \min\{\delta_1, \delta_2\}$. Now let $Y \in \mathbb{R}^2$ satisfy $|X - Y| < \delta$. Then $|X - Y| < \delta_1$, which means that $Y \in B_R(0)$ (so f is well defined at Y). Moreover, we must have $|X - Y| < \delta_2$, which means that $f(X)/2 < f(Y)$. Since we chose Y arbitrarily, we can conclude that $B_\delta(X) \subset B_R(0)$ and that for all $Y \in B_\delta(X)$, $f(Y) > f(X)/2$. Since f is integrable on $B_R(0)$, it must be integrable on $B_\delta(X)$ and have

$$\begin{aligned}\int_{B_\delta(X)} f &\geq \int_{B_\delta(X)} \frac{f(X)}{2} dt \\ &= \frac{f(X)}{2} \int_{B_\delta(X)} dt \\ &> 0\end{aligned}$$

And thus, by the domain splitting properties of integrals, and the fact that f is strictly positive,

$$\begin{aligned}\int_{B_R(0)} f &= \int_{B_\delta(X)} f + \int_{B_R(0)-B_\delta X} f \\ &> \int_{B_R(0)-B_\delta(X)} f \\ &\geq 0\end{aligned}$$

Therefore,

$$\int_{B_R(0)} f > 0$$

■

Using Lemma 1.1, we can establish the following basic, but important, result:

Theorem 1.2: If $u \in \mathfrak{A}_{B_R(0)}$, then $F(u) > 0$.

Proof :

By definition of the absolute value, we must have $|\nabla u(X)|^2 \geq 0$, which means that for all $X \in B_R(0)$,

$$0 < \frac{1}{1 + |\nabla u(X)|^2} \leq 1$$

In particular, this means that $0 < 1/(1 + |\nabla u(X)|^2)$. Since $u \in \mathfrak{A}_{B_R(0)}$, $1/(1 + |\nabla u(X)|^2)$ exists and is continuous almost everywhere in $B_R(0)$. Therefore, by Lemma 1.1,

$$\int_{B_R(0)} u > 0$$

■

Importantly, this means that 0 is a lower bound of $\{F(u) \mid u \in \mathfrak{A}_{B_R(0)}\}$. Therefore,

$$0 \leq \inf\{F(u) \mid u \in \mathfrak{A}_{B_R(0)}\} \tag{1.12}$$

1.5 The Minimum Resistance Problem with Constraints

1.5.1 A Sequence of Spikes

Theorem 1.2 tells us that if u solves (1.9), then we must have $F(u) > 0$. Unfortunately, as the following example shows, the $\inf\{F(u) \mid u \in \mathfrak{A}_{B_R(0)}\} = 0$, which means that (1.9) has no solution. Consider the sequence of functions $\{u_n\}_{n\in\mathbb{R}}$ on $\overline{B_R(0)}$ defined by,

$$u_n(x) = n(R - |x|)$$

Importantly, each of these functions are elements of $\mathfrak{A}_{B_R(0)}$. Some of these functions are depicted in figure 1.5.

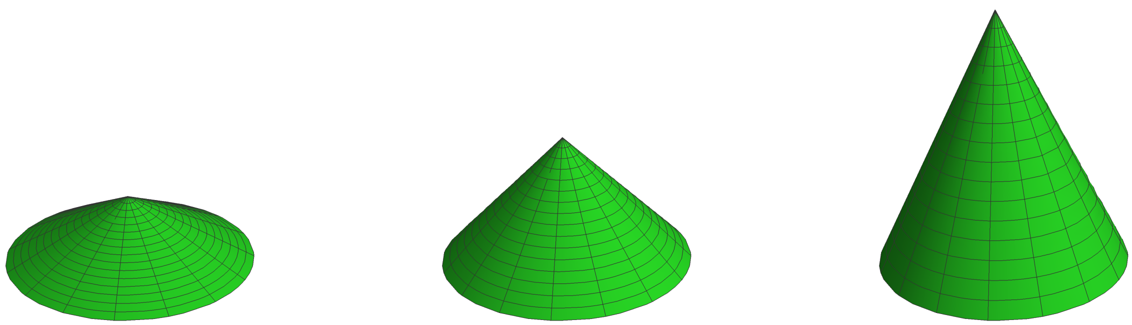

Figure 1.5: The spike functions u_1, u_2, and u_4

Lemma 1.3: $\lim_{n\to\infty} F(u_n) = 0$

Proof :

To begin, let $n \in \mathbb{N}$. By definition, u_n is a function of only $|x|$. Thus, in polar coordinates u_n is a function of r only. In particular, $u(r, \theta) = n(R - r)$. Therefore, by changing F to polar coordinates,

$$F(u_n) = 2\pi \int_0^R \frac{r}{(1 + |u_n'(r)|^2)} dr$$

And, by definition of u_n,

$$u_n'(r) = n$$

Thus,

$$\begin{aligned} F(u_n) &= 2\pi \int_0^R \frac{r}{(1+|u_n'(r)|^2)} dr \\ &= 2\pi \int_0^R \frac{r}{(1+n^2)} dr = \frac{2\pi}{1+n^2} \int_0^R r dr \\ &= \frac{\pi R^2}{1+n^2} \end{aligned}$$

Since R and π are constants, $\lim_{n\to\infty} F(u_n) = \lim_{n\to\infty} \pi R^2/(1+n^2) = 0$. ■

This result tells us that for sufficiently large n, we can make $F(u_n)$ arbitrarily close to 0. Importantly, this means that $\inf\{F(v) \mid v \in \mathfrak{A}_{B_R(0)}\} \leq 0$. Combining this with (1.12), we can conclude that

$$\inf\{F(v) \mid v \in \mathfrak{A}_{B_R(0)}\} = 0 \tag{1.13}$$

Theorem 1.4: The unconstrained Minimum Resistance Problem, (1.9), has no solution. In other words, $\min\{F(v) \mid v \in \mathfrak{A}_{B_R(0)}\}$ does not exist.

Proof :

By (1.13), $\inf\{F(v) \mid v \in \mathfrak{A}_{B_R(0)}\} = 0$. However, by Theorem 1.2, we know that for each $u \in \mathfrak{A}_{B_R(0)}$, $F(u) > 0$. Therefore, each function in $\mathfrak{A}_{B_R(0)}$ has a resistance greater than $\inf\{F(v) \mid v \in \mathfrak{A}_{B_R(0)}\}$, which means that (1.9) has no solution. ■

Notwithstanding, we can still say a lot about the unconstrained variant of the Minimum Resistance Problem. We know that the infimum of resistances for functions in $\mathfrak{A}_{B_R(0)}$ is 0.

We also know that given any $\epsilon > 0$, there is some $N \in \mathbb{R}$ such that for $n \geq N$, $F(u_n) < \epsilon$. In other words, we can think of the functions in the sequence $\{u_n\}$ as "approximate solutions".

1.5.2 Legendre's $\sin^2$ Sequence

It is worth noting that functions in $\{u_n\}$ form a sequence of increasingly tall "spikes" centered at the origin. As the spikes get taller, their gradients get arbitrarily large, which causes the resistance to approach 0. However, the Minimum Resistance Problem was motivated by real-world considerations, namely trying to minimize drag on a ship. With this in mind, it seems appropriate to only consider solutions of bounded height. Let $H > 0$ and let $\mathfrak{A}_{B_R(0),H}$ be defined by

$$\mathfrak{A}_{B_R(0),H} = \{u : B_R(0) \to [0, H] \mid u \text{ is piecewise smooth}\} \tag{1.14}$$

Given this, we can state the "height limited" variant of the Minimum Resistance Problem:

Find the $u \in \mathfrak{A}_{B_R(0),H}$ which gives the least resistance. In other words, find

$$\min\{F(u) \mid u \in \mathfrak{A}_{B_R(0),H}\} \tag{1.15}$$

Notice that $\mathfrak{A}_{B_R(0),H} \subset \mathfrak{A}_{B_R(0)}$. Therefore, Lemma 1.1 is applicable to functions in $\mathfrak{A}_{B_R(0),H}$. Thus, we must have

$$\inf\{F(u) \mid u \in \mathfrak{A}_{B_R(0),H}\} \geq 0 \tag{1.16}$$

Unfortunately, just as with the unconstrained variant, the height-limited variant of the Minimum Resistance Problem has no solution. The counterexample, in this case, is originally due to Legendre (see [Leg86]). Consider the sequence of functions $\{v_n\}_{n\in\mathbb{N}}$ defined by,

$$v_n(x) = H\sin^2\left(\frac{\pi n|x|}{R}\right)$$

Notice that each v_n is an element of $\mathfrak{A}_{B_R(0),H}$ A few of these functions are depicted in Figure 1.6.

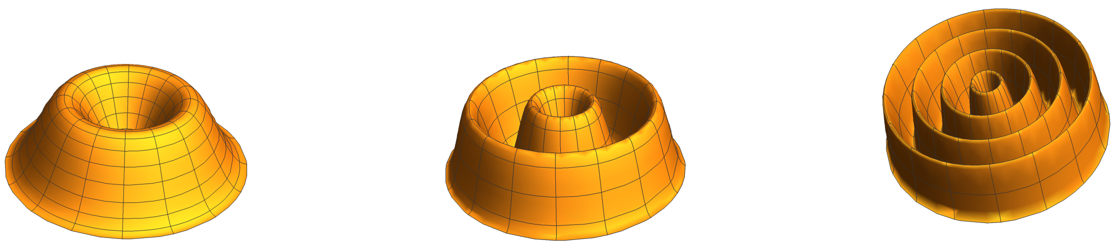

Figure 1.6: Legendre's functions v_1, v_2, and v_4

Lemma 1.5: For each $n \in \mathbb{N}$,

$$F(v_u) \leq 2\pi R\int_0^R \frac{1}{1+v_n'(r)^2}dr \tag{1.17}$$

$$v_n'(r)^2 = \left(\frac{\pi n H}{R}\right)^2 \sin^2\left(\frac{2\pi n r}{R}\right) \tag{1.18}$$

Proof :

To begin, let $n \in \mathbb{N}$. By definition, we can see that v_n is a function of just $|x|$. Thus, by converting to polar coordinates,

$$\begin{aligned} F(v_n) &= \int_{B_R(0)} \frac{1}{1+|\nabla v|^2} \\ &= 2\pi\int_0^R \frac{r}{1+v_n'(r)^2}dr \end{aligned}$$

Importantly, the integrand of $F(v_n)$ is strictly positive. Thus, for each $r \in [0, R]$,

$$\frac{r}{1+v_n'(r)^2} \leq \frac{R}{1+v_n'(r)^2}$$

which means that

$$F(v_n) \leq 2\pi R \int_0^R \frac{1}{1+v_n'(r)^2} dr$$

Which shows (1.17). Now, by definition of v_n,

$$\begin{aligned} v_n'(r) &= \frac{d}{dr} H \sin^2\left(\frac{\pi n r}{R}\right) \\ &= \frac{2Hn\pi}{R} \sin\left(\frac{\pi n r}{R}\right) \cos\left(\frac{\pi n r}{R}\right) \\ &= \frac{Hn\pi}{R} \sin\left(\frac{2\pi n r}{R}\right) \end{aligned}$$

And thus,

$$v_n'(r)^2 = \left(\frac{Hn\pi}{R}\right)^2 \sin^2\left(\frac{2\pi n r}{R}\right)$$

Which shows (1.18). ■

Lemma 1.6: Let $n \in \mathbb{N}$ with $n > 2$. Let p_n denote the proportion of a period of $\sin^2(nx)$ for which $\sin^2(nx) < 1/n$. Then $0 \leq p_n < (2/\pi)(1/\sqrt{n} + 1/n)$. In particular, $\lim_{n\to\infty} p_n = 0$.

Proof :

To begin, since $\sin^2(nx)$ is periodic with period π, we can assume without loss of generality that $-\pi/2 \leq nx \leq \pi/2$. This assumption will simplify the arguments that follow. Suppose that $\sin^2(nx) < 1/n$. Then,

$$-\frac{1}{\sqrt{n}} \leq \sin(nx) \leq \frac{1}{\sqrt{n}}$$

Because of how we restricted x, $\arcsin(\sin(nx)) = nx$. Moreover, since $n \geq 2$, $\arcsin(-1/\sqrt{n})$ and $\arcsin(1/\sqrt{n})$ are well defined. Since arcsin is a strictly increasing function, we must have

$$\arcsin\left(-\frac{1}{\sqrt{n}}\right) \leq nx \leq \arcsin\left(\frac{1}{\sqrt{n}}\right) \tag{1.19}$$

Importantly, since $n \geq 2$, we must have $1/\sqrt{n} < 1/\sqrt{2} < 1$. Since arcsin is twice differentiable in $(-1, 1)$, we can conclude by Taylor's Theorem that given any $t \in (-1, 1)$ with $t \neq 0$, there exists some c_t between 0 and t such that

$$\begin{aligned}\arcsin(t) &= \arcsin(0) + \arcsin'(0)t + \frac{\arcsin''(c_t)}{2}t^2 \\ &= t + \frac{\arcsin''(c_t)}{2}t^2\end{aligned}$$

In particular, this means that there exist some $c_1 \in (-1/\sqrt{n}, 0)$ and $c_2 \in (0, 1/\sqrt{n})$ such that

$$\begin{aligned}\arcsin\left(-\frac{1}{\sqrt{n}}\right) &= -\frac{1}{\sqrt{n}} + \left(\frac{\arcsin''(c_1)}{2}\right)\frac{1}{n} \\ \arcsin\left(\frac{1}{\sqrt{n}}\right) &= \frac{1}{\sqrt{n}} + \left(\frac{\arcsin''(c_2)}{2}\right)\frac{1}{n}\end{aligned}$$

We know that $\arcsin''(x) = x/(1-x^2)^{3/2}$, which is a strictly increasing function. Thus, since $n \leq 2$, we must have $-2 = \arcsin''(-1/\sqrt{2}) < \arcsin''(c_1)$ and $\arcsin''(c_2) < \arcsin''(1/\sqrt{2}) = 2$. Therefore,

$$\begin{aligned}\arcsin\left(-\frac{1}{\sqrt{n}}\right) &> -\frac{1}{\sqrt{n}} - \frac{1}{n} \\ \arcsin\left(\frac{1}{\sqrt{n}}\right) &< \frac{1}{\sqrt{n}} + \frac{1}{n}\end{aligned}$$

Plugging these results (1.19) gives,

$$-\frac{1}{\sqrt{n}} - \frac{1}{n} \leq nx \leq \frac{1}{\sqrt{n}} + \frac{1}{n}$$

This tells us that the set of $nx \in [-\pi/2, \pi/2]$ for which $\sin(nx) < 1/n$ fits within an interval of width $2(1/\sqrt{n} + 1/n)$. Thus, p_n satisfies

$$0 \leq p_n \leq \frac{2}{\pi}(\frac{1}{\sqrt{n}} + \frac{1}{n})$$

We know that $\{2\pi(1/n + 1/\sqrt{n})\}$ converges to 0. Thus, by the Squeeze Theorem, $\lim_{n\to\infty} p_n = 0$. ■

Theorem 1.7: $\lim_{n\to\infty}(F(v_n)) = 0$

Proof :

Let $n \in \mathbb{N}$ with $n > 1$. By Lemma 1.5,

$$F(v_u) \leq 2\pi R \int_0^R \frac{1}{1 + v_n'(r)^2} dr$$

For each $r \in [0, R]$, either $\sin^2(2\pi nr/R) < 1/n$ or not. To aid in the following argument, let

$$S_{<\frac{1}{n}} = \left\{ r \in [0, R] : \sin^2\left(\frac{2\pi nr}{R}\right) < 1/n \right\}$$
$$S_{\geq\frac{1}{n}} = \left\{ r \in [0, R] : \sin^2\left(\frac{2\pi nr}{R}\right) \geq 1/n \right\}$$

By additivity of integrals,

$$F(v_n) \leq 2\pi R \left(\int_{r \in S_{<\frac{1}{n}}} \frac{1}{1 + v_n'(r)^2} dr + \int_{r \in S_{\geq\frac{1}{n}}} \frac{1}{1 + v_n'(r)^2} dr \right) \tag{1.20}$$

Let's focus on the first part of the right side of (1.20). By Lemma 1.6, we know that the proportion of one period of $\sin^2(2\pi r/R)$ for which $\sin^2(2\pi r/R) < 1/n$, denoted p_n, satisfies $0 < p_n < (2/\pi)(1/\sqrt{n} + 1/n)$. As r varies from 0 to R, $\sin^2(2\pi nr/R)$ completes $2n$ periods. Thus, the proportion of $[0, R]$ for which $\sin^2(2\pi r/R) < 1/n$ is also p_n. Therefore, the measure of $S_{<1/n}$, which we will denote by $\mu(S_{<1/n})$, is Rp_n and must be less than $(2R/\pi)(1/\sqrt{n} + 1/n)$. Further, notice that if $r \in S_{<1/n}$ then $1/(1 + u_n'(r)^2) \leq 1$. Therefore, by monotonicity of integration,

$$
\begin{aligned}
0 < \int_{r \in S_{<\frac{1}{n}}} \frac{1}{1 + u_n'(r)^2} &\leq \int_{r \in S_{<\frac{1}{n}}} \\
&= \mu(S_{<\frac{1}{n}}) \\
&< \frac{2R}{\pi}\left(\frac{1}{\sqrt{n}} + \frac{1}{n}\right)
\end{aligned}
$$

Since $\lim_{n\to\infty}(2R/\pi)(1/\sqrt{n} + 1/n) = 0$, by the Squeeze Theorem, we can conclude that,

$$\lim_{n\to\infty} \int_{r \in S_{<\frac{1}{n}}} \frac{1}{1 + u_n'(r)^2} = 0$$

Thus, the first part of the right side of (1.20) converges to 0 as $n \to \infty$.

Now let's focus on $S_{\geq 1/n}$. Since $S_{\geq 1/n} \subset [0, R]$, the measure of $S_{\geq 1/n}$, which we will denoted by $\mu(S_{\geq 1/n})$, must be less than R. Further, if $r \in S_{\geq 1/n}$ then, $\sin^2(2\pi nr/R) \geq 1/n$, which means that $n \geq 1/\sin^2(2\pi nr/R)$. With this, and Lemma 1.5, we can conclude that,

$$\begin{aligned}
\frac{1}{1+u_n'(r)^2} &= \frac{1}{1+\left(\frac{\pi n H}{R}\right)^2 \sin^2(\frac{2\pi n r}{R})} \\
&\leq \frac{1}{\left(\frac{\pi n H}{R}\right)^2 \sin^2(\frac{2\pi n r}{R})} \\
&\leq \left(\frac{R^2}{\pi^2 H^2 n^2}\right) n \\
&= \left(\frac{R^2}{\pi^2 H^2}\right) \frac{1}{n}
\end{aligned}$$

And thus,

$$\begin{aligned}
0 < \int_{r\in S_{\geq \frac{1}{n}}} \frac{1}{1+u_n'(r)^2} dr &\leq \int_{r\in S_{\geq \frac{1}{n}}} \frac{R^2}{\pi^2 H^2 n} dr \\
&= \frac{R^2}{\pi^2 H^2 n} \int_{r\in S_{\geq \frac{1}{n}}} dr \\
&= \frac{R^2}{\pi^2 H^2 n} \mu(S_{\geq \frac{1}{n}}) \\
&\leq \frac{R^3}{\pi^2 H^2 n}
\end{aligned}$$

Since $R^3/(\pi^2 H^2 n) \to 0$ as $n \to \infty$, by the Squeeze Theorem, we can conclude that

$$\lim_{n\to\infty} \int_{r\in S_{\geq \frac{1}{n}}} \frac{1}{1+v_n'(r)^2} = 0$$

Therefore, both parts of the right side of (1.20) converge to 0, which means that

$$\lim_{n\to\infty} 2\pi R \int_0^R \frac{1}{1+v_n'(r)^2} dr = 0$$

Therefore, by the Squeeze Theorem, we can conclude that

$$\lim_{n\to\infty} F(v_n) = 0$$

As claimed. ∎

Therefore, we must have

$$\inf\{F(u) \mid u \in \mathfrak{A}_{B_R(0),H}\} \leq 0$$

Combining this with (1.16), we can conclude that

$$\inf\{F(u) \mid u \in \mathfrak{A}_{B_R(0),H}\} = 0 \tag{1.21}$$

Using the same logic that we used to prove Theorem 1.4 (but with $\{v_n\}_{n\in\mathbb{N}}$ in the place of $\{u_n\}_{n\in\mathbb{N}}$), we can conclude that,

Theorem 1.8: The height limited Minimum Resistance Problem, (1.15), has no solution. In other words, $\min\{F(v) \mid v \in \mathfrak{A}_{B_R(0),H}\}$ does not exist.

The proof of Theorem 1.7 relied on the fact that for large n, $1 + |\nabla v_n|^2$ is sufficiently large for all but an arbitrarily small subset of the domain of integration. Thus, the critical feature of the v_n's that allows $F(v_n)$ to approach 0 is that they oscillate very quickly. With that said, if we look back at how we derived F, then we can see that Legendre's counterexample is, in a sense, antithetical to the spirit of the problem. In particular, we derived (1.7) assuming that each fluid particle collides with the surface of B_{v_n} once. However, looking back at figure 1.6 it is clear that if fluid a particle struck B_{v_n}, then it would probably become trapped B_{v_n}'s rings and collide with B_{v_n} multiple times. In other words, Legendre's bodies fail to satisfy the Single Impact Assumption, which we we used to derive (1.7) in the first place.

With this in mind, it seems logical to consider variants of the Minimum Resistance Problem which satisfy Single Impact Assumption. To do that, however, we need to come up with a set of criteria which guarantee the Single Impact Assumption. This is the subject of the next section.

1.6 Single Impact Conditions

1.6.1 The Single Impact Condition

To derive a general condition for the Single Impact Assumption, we need to return to the elastic fluid model with which we derived the functional F. In particular, we need to study the trajectory of fluid particles after they collide with the body. This development will lean heavily on results derived in section 1.2.2 (where we derived the functional F). Thus, the reader should familiarize themselves with that section before reading this one. The general Single Impact Condition was first formulated by G. Buttazzo, [BFK95]. The derivation that follows was inspired by section 5 of that paper.

Let $u : \overline{B_R(0)} \to [0, H]$ and suppose that a particle impacts the body B_u at the point $p_0 = (x_0, y_0, u(x_0, y_0))$. We will assume that u is differentiable at (x_0, y_0). We will use the (x, y, z) coordinate system defined in section 1.2.2. In that section, we showed that after a particle collides with B_u, its velocity is given by (1.6), which we restate here for convenience:

$$\overrightarrow{v_f} = |\overrightarrow{v_i}| \frac{1}{1 + |\nabla u(x_0, y_0)|^2} \left(-2\nabla u(x_0, y_0) + \left(1 - |\nabla u(x_0, y_0)|^2\right) \hat{e_z}\right)$$

Thus, the particle's final velocity has a component in the direction of $\nabla u(x_0, y_0)$ and a component in the direction of $\hat{e_z}$. In particular, the particle's motion in the xy plane will be in the direction of $-\nabla u(x_0, y_0)$. However, because of how we defined the (x, y, z) coordinate system, the (x, y) component of the particle's motion is in the direction of the x axis. Therefore, $\hat{e_x} = -\nabla u(\hat{x_0}, y_0)$, where $\nabla u(\hat{x_0}, y_0)$ is the unit vector in the direction $\nabla u(x_0, y_0)$. This means that the particle travels in the xz plane. Therefore,

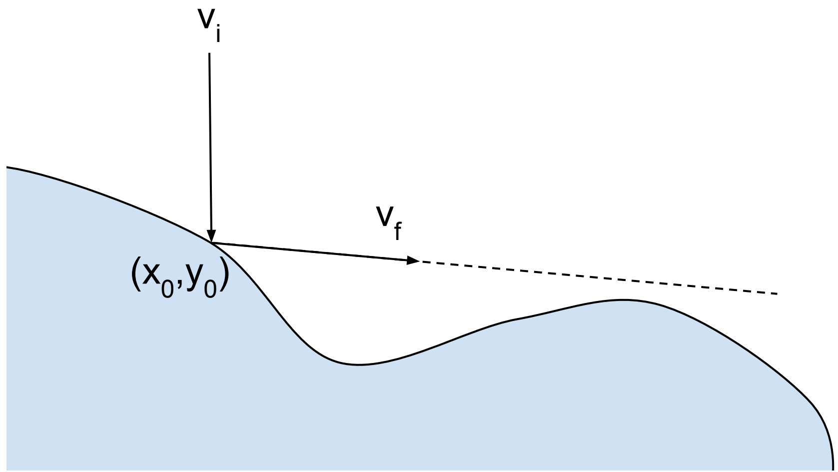

Figure 1.7: The particle's trajectory in the xz plane.

$$\overrightarrow{v_f} = |\overrightarrow{v_i}|\frac{1}{1+|\nabla u(x_0,y_0)|^2}\left(2|\nabla u(x_0,y_0)|\hat{e_x} + \left(1-|\nabla u(x_0,y_0)|^2\right)\hat{e_z}\right)$$

Let s_f denote the slope of the particle's final velocity in the xz plane,

$$\begin{aligned} s_f &= \frac{1-|\nabla u(x_0,y_0)|^2}{2|\nabla u(x_0,y_0)|} \\ &= \frac{1}{2}\left(\frac{1}{|\nabla u(x_0,y_0)|} - |\nabla u(x_0,y_0)|\right) \end{aligned} \tag{1.22}$$

With this established, we can parameterize the x and z components of the particle's post-impact trajectory as follows (for $t > 0$): [§]

$$x(t) = x_0 + t \tag{1.23}$$

$$z(t) = u(x_0,y_0) + s_f t \tag{1.24}$$

[§]the t in this equation does not refer to actual time.

We are now ready to derive the Single Impact Condition. As long as the particle's post-impact trajectory remains above B_u, it will not collide with B_u again. More specifically, the Single Impact Assumption is satisfied if

$$u((x_0 + t, y_0)) \leq z(t) = u(x_0, y_0) + s_f t \tag{1.25}$$

Notice, however, that $(x_0 + t, y_0) = (x_0, y_0) + (t, 0)$. Using the fact that $\hat{e_x} = -\nabla u(\hat{x_0}, y_0) = \nabla u / |\nabla u(x_0, y_0)|$, we can conclude that

$$(x_0 + t, y_0) = (x_0, y_0) - t\nabla u(x_0, y_0)/|\nabla u(x_0, y_0)|$$

Substituting this into (1.25) gives

$$u\left((x_0, y_0) - t\frac{\nabla u(x_0, y_0)}{|\nabla u(x_0, y_0)|}\right) \leq u(x_0, y_0) + s_f t$$

If we replace t with $t|\nabla u(x_0, y_0)|$, then we have

$$u((x_0, y_) - t\nabla u(x_0, y_0)) \leq u(x_0, y_0) + s_f t|\nabla u(x_0, y_0)| \tag{1.26}$$

Substituting in (1.22) for s_f and rearranging gives

$$\frac{u((x_0, y_0) - t\nabla u(x_0, y_0)) - u(x_0, y_0)}{t} \leq \frac{1}{2}\left(1 - |\nabla u(x_0, y_0)|^2\right) \tag{1.27}$$

Given this, we can conclude that a body B_u satisfies the Single Impact Assumption if given any $(x_0, y_0) \in B_R(0)$ such that u is differentiable at (x_0, y_0), (1.27) holds for all $t > 0$ such that $(x_0, y_0) - t\nabla u(x_0, y_0) \in \overline{B_R(0)}$. Therefore, (1.27) is called the Single Impact Condition.

Let $\mathfrak{S}_{B_R(0),H}$ be defined by

$$\mathfrak{S}_{B_R(0),H} = \{u \in \mathfrak{A}_{B_R(0),H} \mid u \text{ satisfies the Single Impact Condition}\}$$

1.6.2 Concavity

In this subsection, we will show that all convex bodies satisfy the Single Impact Condition, and therefore satisfy the Single Impact Assumption. Let $\mathfrak{C}_{B_R(0),H}$ be defined by

$$\mathfrak{C}_{B_R(0),H} = \{u : B_R(0) \to [0, H] \mid u \text{ is concave and piecewise smooth}\}$$

Given this, we can state the "concave" variant of the Minimum Resistance Problem: :

Find the $u \in \mathfrak{C}_{B_R(0),H}$ which has the lowest resistance. In other words, find

$$\min\{F(u) \mid u \in \mathfrak{C}_{B_R(0),H}\} \tag{1.28}$$

Unlike the two variants of the Minimum Resistance Problem that we have looked at thus far, the concave variant of the Minimum Resistance Problem (1.28) has a solution. The interested reader can find a proof in section 2 of [BFK95].

Plakhov, [Pla19], considered the Minimum Resistance Problem when Ω is strictly convex (rather than a circle) and the admissible functions $u : \Omega \to [0, H]$ are concave. He showed that if $\partial\Omega$ satisfies certain assumptions, then any optimal solution u must be zero along the $\partial\Omega$.

Every convex body satisfies the Single Impact Condition. To see this, suppose that $u \in \mathfrak{C}_{B_R(0),H}$. Then B_u is convex. If a particle strikes u at p_0, then its final trajectory will move along the line described by equations (1.23) and (1.24). We know that, in the xz plane, the slope of u at (x_0, y_0) is $-\nabla u(x_0, y_0) \cdot e_x$. However, we also know that $e_x = -\nabla u(\hat{x_0}, y_0)$, which means that the slope of u in the xz plane at (x_0, y_0) is $-|\nabla u(x_0, y_0)|$. However, we know that $s_f > -|\nabla u(x_0, y_0)|$. This tells us that the for some subset of $t < 0$, the trajectory line defined by equations (1.23) and (1.24) is below u, while for some subset of $t > 0$, the trajectory is above u. This can be see in figure 1.7. Thus, for some $t_1 < 0$, the trajectory line is inside B_u. If the particle collided with the body again, then the trajectory line would have to pass through B_u for some $t_2 > 0$. But, since B_u is convex, the trajectory line must be contained in B_u for all $t \in (t_1, t_2)$. This, however, is impossible, since we know that the particle's trajectory lies above u, and therefore outside of B_u right after the impact. Therefore, u satisfies the Single Impact Condition, and therefore satisfies the Single Impact Assumption. Thus,

$$\mathfrak{C}_{B_R(0),H} \subset \mathfrak{S}_{B_R(0),H} \subset \mathfrak{A}_{B_R(0),H} \subset \mathfrak{A}_{B_R(0),H}$$

The converse is not true, however. In particular, suppose that $u \in \mathfrak{S}_{B_R(0),H}$ has the property that the normal vector to ∂B_u (assuming this vector is defined) makes an angle of less than $\pi/6$ with the particle's initial velocity. The graph of such a function could, for example, resemble shallow waves, and therefore fail to be concave. However, particles reflected from the surface would do so with an angle of $< \pi/3$ relative to their incident velocity, which means that their trajectory forms an angle of $> \pi/6$ with the xy plane. Thus, the particle's paths must remain above B_u, which means that u satisfies the Single Impact Condition.

Chapter 2

The Radial Case

Rotational symmetry is inherent to the Minimum Resistance Problem. In particular, the domain of interest, $\overline{B_R(0)}$, is rotationally symmetric. Further, in both variants of the Minimum Resistance Problem that we studied in the last chapter, the approximate solutions were rotationally symmetric. Therefore, it seems logical that the solution to the Concave variant of the Minimum Resistance Problem, (1.28), should also be rotationally symmetric. Newton assumed this and solved (1.28) on the subset of concave bodies that are rotationally symmetric. The rotationally symmetric case of the concave variant and Newton's solution to it are the subjects of this chapter.

In *Philosophiæ Naturalis Principia Mathematica*, Newton studied two closely related variants of the Minimum Resistance Problem. The first was to find the optimal conical frustum. Newton solved this variant using the theory of minima and maxima of functions. The second was to find the optimal rotationally symmetric convex body (which we will refer to as the "radial case"). As we will see, this variant yields an unexpected solution.

2.1 The Radial Case

Before we study either variant, we need to define the admissible functions and formulate the radial case of the Minimum Resistance Problem. To begin, let $\mathfrak{R}_{B_R(0),H}$ be defined by

$$\mathfrak{R}_{B_R(0),H} = \{u : [0,R] \to [0,H] \mid u \text{ is piecewise smooth, } u' \text{ is decreasing, and } u'(0) \leq 0\}$$

For each $u \in \mathfrak{R}_{B_R(0),H}$, let B_u denote the body generated by rotating u around the z axis.* Importantly, if $u \in \mathfrak{R}_{B_R(0),H}$, then B_u must be convex.† Further, in chapter 1, we saw that if u is rotationally symmetric then,

$$F(u) = 2\pi \int_0^R \frac{r}{1+u'(r)^2} dr \tag{2.1}$$

Given this, we can state the radial case of the Minimum Resistance Problem:

> Find the $u \in \mathfrak{R}_{B_R(0),H}$ which has the lowest resistance. That is, find
>
> $$\min\{F(u) \mid u \in \mathfrak{R}_{B_R(0),H}\} \tag{2.2}$$

Interestingly, as the next theorem shows, we can solve (2.2) by considering only the functions $u \in \mathfrak{R}_{B_R(0),H}$ that have $u(0) = H$ and $u(R) = 0$.

Theorem 2.1: If $u \in \mathfrak{R}_{B_R(0),H}$ has either $u(0) \neq H$ or $u(R) \neq 0$ then there is some $\epsilon > 0$ such that the function $w_\epsilon(r) = (1+\epsilon)(u(r) - u(R))$ is in $\mathfrak{R}_{B_R(0),H}$ and has $F(w_\epsilon) < F(u)$.

Proof :

---

*This is the same notation that we used in chapter 1. In general, B_u means "the body associated with u". Whether the body associated with u is he body generated by rotating u around the z axis or the region between $B_R(0)$ and the graph of u should be apparent with context.

†A differentiable function of one variable is convex if and only if its derivative is decreasing. If u' is decreasing, then $u'(0)$ is the maximum slope. If $u'(0)$ is positive, then there will be a depression at the top of B_u, which violates convexity. Thus, we also need $u'(0) \leq 0$.

To begin, let $\epsilon > 0$. Then, $w'_\epsilon(r) = (1+\epsilon)u'(r)$, which means that for all $r \in [0, R]$, $r/(1 + w'_\epsilon(r)^2) < r/(1 + u'(r)^2)$. Thus, $F(w_\epsilon) < F(u)$. Therefore, our task is to show that that if u fails to have either $u(0) = R$ or $u(R) = 0$, then there exists some ϵ such that $w_\epsilon(r)$ is admissible (an element of $\mathfrak{R}_{B_R(0),H}$). Importantly, given any $\epsilon > 0$, since u' is decreasing, w'_ϵ is also decreasing. Similarly, for all ϵ, $w'_\epsilon(0) \leq 0$ since $u'(0) \leq 0$. Thus, if we can find an $\epsilon > 0$ such that the range of w_ϵ is contained in $[0, H]$, then w_ϵ must be admissible. Suppose that $u(R) \neq 0$. The $u(0) \neq H$ case is similar. Since u is non-negative, we must have $u(R) > 0$. Thus, $u(0) - u(R) < H$. Now, pick any $\epsilon > 0$ such that $(1 + \epsilon)(u(0) - u(R)) \leq H$. Then $w_\epsilon(0) \leq H$ and $w_\epsilon(R) = (1 + \epsilon)(u(R) - u(R)) = 0$, which means that the range of w_ϵ is contained in $[0, H]$. Therefore, w_ϵ is admissible and, by the argument at the start of this proof, has $F(w_\epsilon) < F(u)$. ■

Theorem 1.2 tells us that if $u \in \mathfrak{R}_{B_R(0),H}$ solves (2.2), then it must have $u(0) = H$ and $u(R) = 0$. Therefore, we are justified in adding the conditions $u(0) = H$ and $u(R) = 0$ to (2.2). To that end, let $\mathfrak{R}'_{B_R(0),H}$ be defined by

$$\mathfrak{R}'_{B_R(0),H} = \{u \in \mathfrak{R}_{B_R(0),H} \mid u(0) = R \text{ and } u(R) = 0\}$$

With the extra constraints, (2.2) becomes the following:

Find the $u \in \mathfrak{R}'_{B_R(0),H}$ which has the lowest resistance. That is, find

$$\min\{F(u) \mid u \in \mathfrak{R}'_{B_R(0),H}\} \tag{2.3}$$

2.2 The Optimal Conical Frustum

In this section, we will find the optimal conical frustum. A conical frustum is the solid that remains when the "tip" of a cone with a circular base is removed. More precisely, a conical frustum is the part of a cone that lies between its base and a plane that is parallel to the base. Every conical frustum has a piecewise decreasing slope and a slope of 0 at $r = 0$. Therefore, every conical frustum is an element of $\mathfrak{R}_{B_R(0),H}$. The optimal conical frustum will give us some insight into the nature of the radial variant of the Minimum Resistance Problem. Newton found the optimal conical frustum using geometric considerations. The interested reader can find a detailed account of his approach in [Gol80]. Here, we will find the optimal conical frustum using basic calculus.

Figure 2.1 depicts a conical frustum. $a \in [0, H]$ is the height of the conical frustum. r is the radius of the top face of the conical frustum. z is the length of the cone (from which the conical frustum comes) beyond the top of the frustum. Thus, $z \in [0, \infty)$

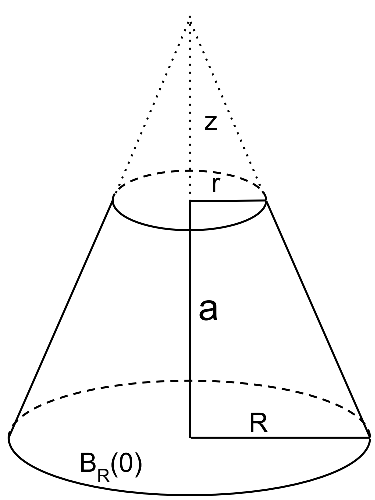

Figure 2.1: A conical frustum.

By inspection,

$$\frac{r}{z} = \frac{R}{z+a}$$

which means that $r = Rz/(z+a)$. Since $a \in [0, H]$ and $z \geq 0$, we must have $r \in [0, R]$. Our goal is to find the values of z and a whose corresponding conical frustum has the least resistance.

Let $u_z : [0, R] \to [0, a]$ defined by

$$u_z(x) = \begin{cases} a & x \in [0, r] \\ (z+a) - \frac{(z+a)x}{R} & x \in [r, R] \end{cases}$$

Then the conical frustum depicted in figure 2.1 is the body generated by rotating u_z about the z axis. Let $Res(z, a)$ denote the resistance of B_{u_z}. Then,

$$\begin{aligned} Res(z,a) &= F(u) \\ &= 2\pi \int_0^R \frac{x}{1+u_z'(x)^2} dx \\ &= 2\pi \left(\int_0^r \frac{x}{1+u_z'(x)^2} dx + \int_r^R \frac{x}{1+u_z'(x)^2} dx \right) \\ &= 2\pi \left(\int_0^r x dx + \int_r^R \frac{x}{1+\left(\frac{z+a}{R}\right)^2} dx \right) \\ &= 2\pi \left(\frac{r^2}{2} + \frac{R^2 - r^2}{2\left(1 + (\frac{z+a}{R})^2\right)} \right) \\ &= \frac{\pi}{R^2 + (z+a)^2} \left(r^2 (z+a)^2 + R^4 \right) \end{aligned}$$

However, since $r = Rz/(z+a)$, we must have $r^2(z+a)^2 = R^2z^2$. Thus,

$$Res(z,a) = \frac{\pi R^2}{R^2 + (z+a)^2}(z^2 + R^2) \tag{2.4}$$

Thus, (2.4) gives the resistance of the conical frustum as a function of z and a. Thus, the resistance is differentiable with respect to both a and z (for $a \in [0, H]$ and $z \geq 0$). In this thesis, we will denote the partial derivative of a function f with respect to a variable x by $D_x f$.

By inspection, $Res(z,a)$ decreases as a increases. Thus, any conical frustum with $a < H$ would have a larger resistance than the corresponding conical frustum with $a = H$. Therefore, the optimal conical frustum must have $a = H$. Thus, to minimize $Res(z,a)$, we only need to minimize it with respect to z. There are three possibilities: either the optimal z occurs at 0, somewhere in $(0, \infty)$, or as z approaches ∞. By inspection, as $z \to \infty$, the corresponding conical frustum approaches a cylinder. This is supported by the fact that as z approaches ∞, $Res(a,z)$ approaches πR^2, which is resistance for a cylinder that we derived in section 1.3.2. Further, when $z = 0$,

$$Res(0,a) = \frac{\pi R^4}{R^2 + a^2} \tag{2.5}$$

Finally, if the optimal z occurs in $(0, \infty)$, then we must have $\mathrm{D}_z\, Res(z,a) = 0$, which means that

$$\mathrm{D}_z\, Res(z,a) = \frac{2\pi R^2 z}{R^2 + (a+z)^2} - \frac{\pi R^2(R^2+z^2)\,(2(a+z))}{\left(R^2 + (a+z)^2\right)^2} = 0 \tag{2.6}$$

Interestingly, this tells us that

$$\mathrm{D}_z \, Res(z,a)\bigg|_{z=0} = -\frac{2\pi R^4 a^2}{R^2+a^2}$$

Which is negative. Therefore, the resistance is decreasing at $z = 0$, which tells us that the optimal conical frustum does not have $z = 0$ and is, therefore, not a cone. Thus, the optimal conical frustum must have $z \in (0, \infty)$. Multiplying (2.6) by $(R^2 + (z + a)^2)^2/(2\pi R^2)$ and solving for z gives,

$$z = \frac{1}{2}\left(-a \pm \sqrt{a^2+4R^2}\right)$$

Since we assumed that $z \geq 0$, the optimal z must be

$$z_{optimal} = \frac{1}{2}\left(-a + \sqrt{a^2+4R^2}\right) \tag{2.7}$$

Thus, (2.4) has just one extremum in $[0, \infty)$. Since we know that the resistance decreases at $z = 0$ and increases as $z \to \infty$, we can conclude that this extremum, the solution to (2.7) is a minima. Therefore, (2.7) gives the z for the optimal conical frustum . The corresponding resistance (with $a = H$) is,

$$Res(z_{optimal}, H) = \frac{\pi R^2\left(\frac{1}{4}\left(\sqrt{H^2+4R^2}-H\right)^2+R^2\right)}{\left(\frac{1}{2}\left(\sqrt{H^2+4R^2}-H\right)+H\right)^2+R^2} \tag{2.8}$$

Figure 2.2 shows optimal conical frustum for $R = 1$ and $H = 1$.

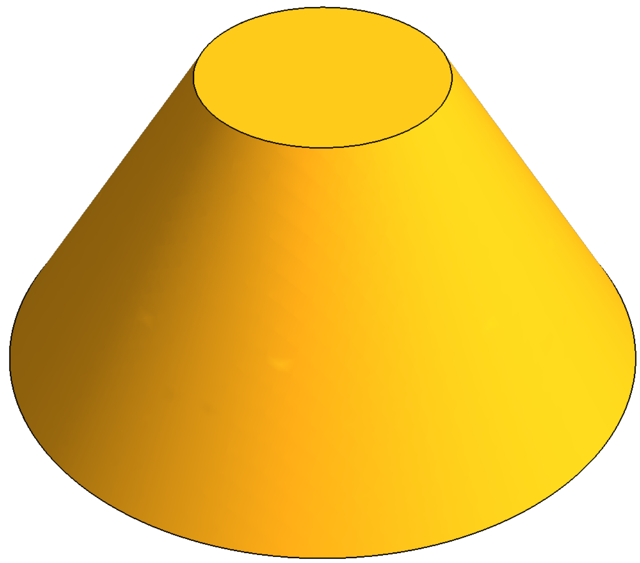

Figure 2.2: The optimal conical frustum for $R = 1$, $H = 1$

The fact that the optimal conical frustum has a blunt end should seem counterintuitive. Intuitively, a cone (which corresponds to $z = 0$) should be more aerodynamic than a conical frustum with a blunt end. A flat surface incurs more resistance per unit cross-sectional area than a curved one (this can be inferred from (2.1)). Notice, however, that a conical frustum (with a blunt end) will have a steeper edge than the cone of the same height. Thus, the fact that the optimal conical frustum is not a cone tells us that the reduction in resistance from the steeper edge more than offsets the gain in resistance from a blunt end. This suggests that the optimal rotationally symmetric convex body may also have a blunt end. In the next section, we will show that this suggestion is correct.

2.3 The Optimal Rotationally Symmetric Body

In this section, we will consider the radial case and Newton's solution to it. Filling in all of the details to derive the optimal rotationally symmetric convex body is, unfortunately, a fairly involved process. Thus, in this section, we will only give a partial derivation of the optimal rotationally symmetric convex body. The interested reader can find a more complete discussion of the derivation in either [Gol80] or [Par62].

To begin, notice that the integrand in F does not depend on u. Therefore, the Euler-Lagrange equation for (2.1) is [‡]

$$\frac{d}{dr}\,\mathrm{D}_{u'}\,F(u'(r),r) = \frac{d}{dr}\left(\frac{-2ru'(r)}{\left(1+u'(r)^2\right)^2}\right) = 0 \tag{2.9}$$

Integrating this with respect to r and letting $2c_1$ be the constant of integration gives

$$\frac{ru'(r)}{\left(1+u'(r)^2\right)^2} = -c_1$$

And thus,

$$ru'(r) = -c_1\left(1+u'(r)^2\right)^2 \tag{2.10}$$

[‡]The Euler-Lagrange equation is the fundamental equation in the Calculus of Variations. The interested reader can find more about the Euler-Lagrange equation in particular and the Calculus of Variations in general in [Par62] or [Kot14]

This gives us a non-linear first-order equation for $u(r)$. In general, either $c_1 < 0$, $c_1 = 0$, or $c_1 > 0$. If $c_1 < 0$, then we must have $u'(r) > 0$, which implies the solution to (2.10) is strictly increasing. We know that this is false, however, since the admissible functions are decreasing. Similarly, if $c_1 = 0$, then $ru'(r) = 0$, which means that $u(r)$ is a constant. If this were the case, however, then B_u would be a cylinder. In the previous section, however, we showed that the optimal conical frustum has lower resistance than a cylinder. Therefore, we must have $c_1 > 0$. This tells us that $u'(r)$ is strictly negative. Therefore, $u'(r)$ is never zero.

Let $p = -u'(r)$. Then p must be strictly positive. Further, notice that substituting p into (2.10) gives,

$$r(p) = c_1 \left(\frac{1}{p} + 2p + p^3 \right) \tag{2.11}$$

This means that r is a function of p. We can use this and the chain rule to express u as a function of p (which is possible because u is a function of r). We know that u is differentiable with respect to r. Further, by (2.11), r is differentiable with respect to p. Therefore,

$$\begin{aligned}
\frac{du}{dp} &= \left(\frac{du}{dr} \right) \left(\frac{dr}{dp} \right) \\
&= c_1 \left(\frac{1}{p} - 2p - 3p^3 \right)
\end{aligned}$$

And thus,

$$u(p) = c_0 + c_1 \left(\ln(p) - p^2 - \frac{3}{4}p^4 \right) \tag{2.12}$$

Together, equations (2.11) and (2.12) define the solution curve, and tell us that the solution is a function of $p = -u(r)$. This is a remarkable result. These equations tell us that each point on the solution curve has a unique slope (with respect to r). Therefore, we can conclude that the optimal conical frustum, which we derived in section 2.1, is not the optimal solution, since the slope of any u corresponding to a conical frustum has just two distinct values.

All that's left is to find c_0 and c_1 such that $u|_{r=0} = H$ and $u|_{r=R} = 0$. Unfortunately, there are no solutions of (2.11) and (2.12) which satisfy these conditions. In particular, since c_1 is positive, and since p is positive, r must be strictly positive. There is no $p \in (0, \infty)$ such that $r(p) = 0$. This seems to suggest that the problem is ill-posed. Luckily, it is possible to resolve this inconsistency, though doing so goes beyond the scope of this thesis. The interested reader can find more about the resolution in either [Gol80] or [Par62].

We can get an idea of where things go wrong by looking at the derivatives of u and r with respect to p,

$$\frac{dr}{dp} = c_1 \left(-\frac{1}{p^2} + 2 + 3p^2 \right) = c_1 \frac{(1+p^2)(3p^2-1)}{p^2}$$
$$\frac{du}{dp} = c_1 \left(\frac{1}{p} - 2p - 3p^3 \right) = -c_1 \frac{(1+p^2)(3p^2-1)}{p}$$

Both of which have a root at

$$p = \frac{1}{\sqrt{3}} \tag{2.13}$$

Which corresponds to the coordinates

$$u = c_0 - c_1 \left(\frac{1}{2} \ln(3) + \frac{5}{12} \right)$$
$$r = c_1 \frac{16\sqrt{3}}{9}$$

In fact, the solution curve has a cusp at this value of p. Figure 2.3 shows $r(p)$ and $u(p)$ when $c_0 = 5$ and $c_1 = .5$

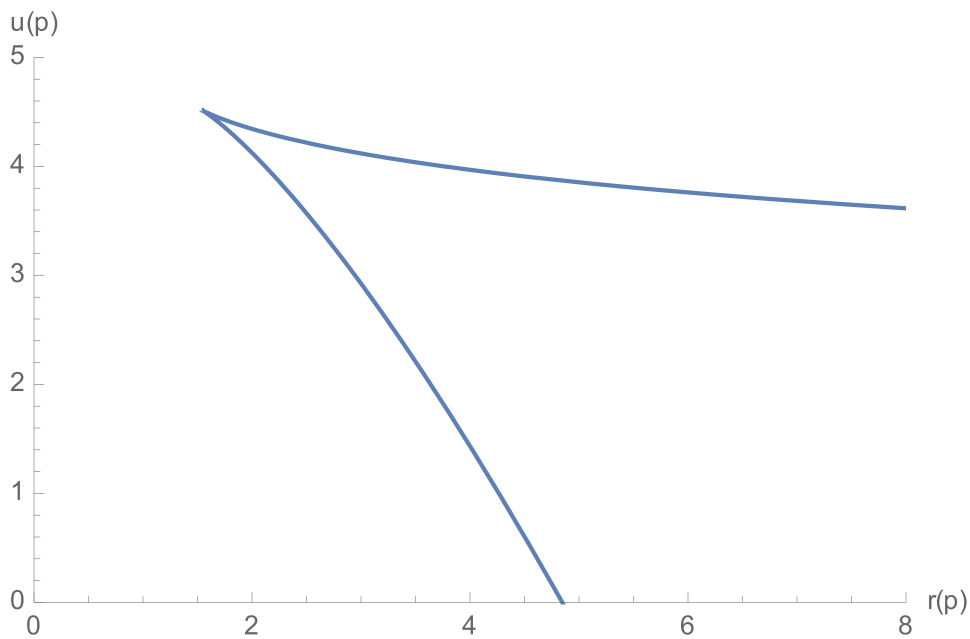

Figure 2.3: Plot of $u(p)$ and $r(p)$ for $c_0 = 5$ and $c_1 = .5$

Figure 2.3 is concerning. It shows that u and r do not have a functional relationship.§ Pars, [Par62], showed that the bottom curve in figure 2.3 (which corresponds to $p \geq 1/\sqrt{3}$) is the optimal curve.

Pars, [Par62], also shows that the optimal solution has $c_0 = -(7/4)c_1$ and consists of a horizontal line from $(0, H)$ to $(4c_1, H)$ appended to the curve described by equations (2.11) and (2.12). The precise value of c_1 is is determined by the conditions $u|_{r=R} = 0$ and $u|_{r=4a} = H$. Figure 2.4 depicts the optimal solution for $H = R/2$, $H = R$, and $H = 2R$.

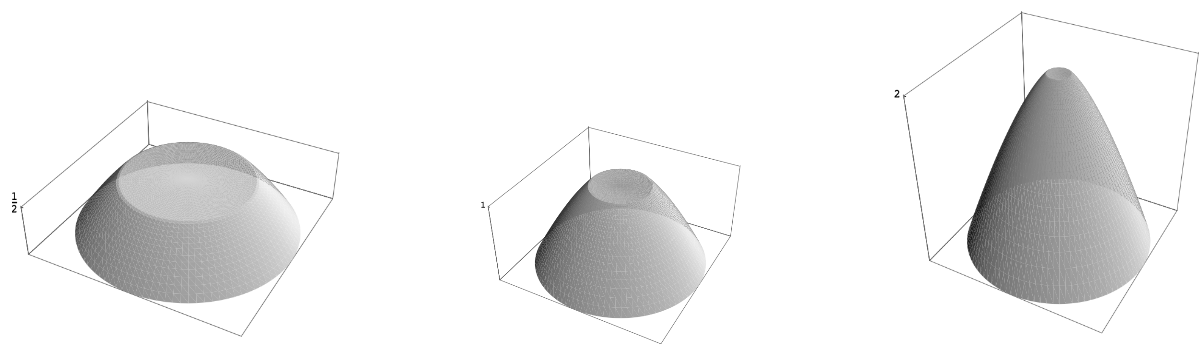

Figure 2.4: Newton's optimal solution for $H = R/2$, $H = R$, and $H = 2R$

2.4 Newton's Solution and the Concave Variant

For hundreds of years, mathematicians assumed that Newton's solution also solved the concave variant of the Minimum Resistance Problem, (1.28), though no one could prove this proposition. In 1995 Paolo Guasoni, however, [Gua95] showed that there are non-rotationally symmetric elements of $\mathfrak{C}_{B_R(0),H}$ which have a lower resistance than Newton's rotationally

§This is a result of fact that we parameterized the solution curve.

symmetric solution. Therefore, the solution to (1.28) is not rotationally symmetric. Paolo's discovery sparked two decades of research on the concave variant. The interested reader can find out more about the solutions to the concave variant in [Wac14]. However, we will not discuss the concave variant any further. Instead, we will turn our attention to the Minimum Resistance Problem over the set of functions that satisfy the Single Impact Assumption, which is the subject of the next chapter.

Chapter 3

The Single Impact Condition Case

In this chapter, we will consider the Minimum Resistance Problem for bodies that satisfy the Single Impact Condition. This variant of the problem has been studied extensively for the past thirty years. During that time, numerous authors have discovered a myriad of properties of the Minimum Resistance Problem and its solutions.

Comte and Lachand-Robert, [CL01b], studied the Minimum Resistance Problem on the set of bodies that are rotationally symmetric and satisfy the Single Impact Condition. This is a generalization of the problem that we considered in chapter 2. They showed that there is a unique solution for sufficiently large H. The solution is similar to Newton's solution, but with the flat cap replaced with a depression. The depression is constructed such that the resulting body satisfies the Single Impact Condition but has lower resistance than Newton's optimal solution. Figure 3, which was borrowed from [CL01b], depicts the solution.

In this chapter, we will study the Minimum Resistance Problem on $\mathfrak{S}_{B_R(0),H}$ (the set of functions from $B_R(0)$ to $[0, H]$ that satisfy the Single Impact Condition). The forthcoming discussion, theorems, and arguments were borrowed from [Pla16]. In that paper, Plakhov allows Ω to be a convex, open, and bounded set. $B_R(0)$ certainly satisfies these conditions. Thus, Plakhov's paper is more general than our discussion will be. However, to keep in

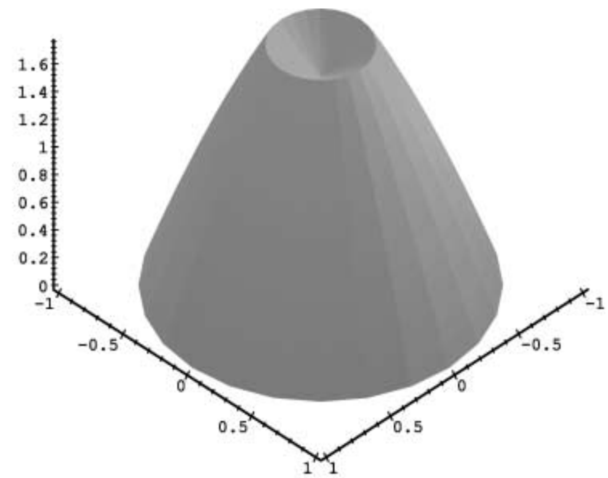

Figure 3.1: The optimal rotationally symmetric body that satisfies the Single Impact Condition.

line with Newton's problem, and to simplify the arguments, we will consider the case when $\Omega = B_R(0)$.

3.1 There is No Optimal Solution

We want to find the $u \in \mathfrak{S}_{B_R(0),H}$ which corresponds to

$$\min\{F(v) : v \in \mathfrak{S}_{B_R(0),H}\} \tag{3.1}$$

Unfortunately, Comte and Lachand-Robert, [CL01b], showed that (3.1) does not have a solution. In particular, they showed that there is no admissible u such that $F(u) = \inf\{F(v) : v \in \mathfrak{S}_{B_R(0),H}\}$. With that said, we know that for all piecewise smooth u, $F(u) \geq 0$. Therefore, the set $\{F(v) : v \in \mathfrak{S}_{B_R(0),H}\}$ is bounded below and, therefore, has a well defined infimum. Thus, rather than finding the minimizer, we will instead find the infimum of $\{F(v) : v \in \mathfrak{S}_{B_R(0),H}\}$. We will also construct a family of functions whose resistance is very close to this infimum.

In the following sections, we will derive a positive lower bound for $\{F(v) : v \in \mathfrak{S}_{B_R(0),H}\}$. We will then then show that this lower bound is the infimum by demonstrating that for any $\epsilon > 0$, we can construct a function $u_\epsilon \in \mathfrak{S}_{B_R(0),H}$ such that $F(u_\epsilon)$ is less than ϵ away from the lower bound.

3.2 A Lower Bound for $F(u)$

For each $X \in B_R(0)$, let $\text{dist}(X, \partial B_R(0))$ denote the distance from X to $\partial B_R(0)$. That is,

$$\begin{aligned} \text{dist}(X, \partial B_R(0)) &= \inf\{|X - Y| : Y \in \partial B_R(0)\} \\ &= R - |X| \end{aligned}$$

Now let us define $\phi\left(B_R(0), H\right)$ as

$$\begin{aligned} \phi\left(B_R(0), H\right) &= \int_{B_R(0)} \frac{1}{2}\left(1 - \frac{H}{\sqrt{H^2 + \text{dist}(X, \partial B_R(0))^2}}\right) dX \qquad (3.2) \\ &= \int_{B_R(0)} \frac{1}{2}\left(1 - \frac{H}{\sqrt{H^2 + (R - |x|)^2}}\right) dX \end{aligned}$$

By changing to polar coordinates,

$$\phi\left(B_R(0), H\right) = \pi \int_0^R \left(1 - \frac{H}{H^2 + (R - r)^2}\right) r dr$$

In this section, we will show that $\phi\left(B_R(0), H\right)$ is a lower bound of $\{F(u) : u \in \mathfrak{S}_{B_R(0),H}\}$. In the next two sections, we will show that it is the infimum of this set.

Theorem 3.1: For all $u \in \mathfrak{S}_{B_R(0),H}$, $F(u) \geq \phi\left(B_R(0), H\right)$.

Proof :

To begin, suppose that a particle collides elastically with B_u at $X \in B_R(0)$. Let $v_{f,3}(X)$ denote the z component of the particle's post-impact velocity. By (1.6), we know that

$$v_{f,3}(X) = |\overrightarrow{v_i}| \frac{1 - |\nabla u(X)|^2}{1 + |\nabla u(X)|^2}$$

And thus,

$$\frac{1}{2|\overrightarrow{v_i}|} \left(1 + v_{f,3}(X)\right) = \frac{1}{1 + |\nabla u(X)|^2}$$

Since we chose X arbitrarily, we can conclude that

$$\begin{aligned} F(u) &= \int_{B_R(0)} \frac{1}{1 + |\nabla u(X)|^2} dX \\ &= \int_{B_R(0)} \frac{1}{2|\overrightarrow{v_i}|} \left(1 + v_{f,3}(X)\right) dX \end{aligned}$$

With that established, let's focus on v_f. Suppose that a particle hits the surface of B_u at $(x, y, u(x, y))$. For brevity, let $X = (x, y)$ After the impact, the particle may either pass through the xy plane or not. Suppose that it does, and let $X' \in \mathbb{R}^2$ denote the point at which the particle crosses the xy plane. Then, since the collision is elastic $(|\overrightarrow{v_i}| = |\overrightarrow{v_f}|)$,

$$\overrightarrow{v_f} = |\overrightarrow{v_i}| \frac{(X' - X - u(X)e_z)}{\sqrt{u(X)^2 + |X - X'|^2}}$$

And thus,

$$v_{f,3}(X) = |\overrightarrow{v_i}| \frac{-u(X)}{\sqrt{u(X)^2 + |X - X'|^2}} \tag{3.3}$$

Importantly, since B_u satisfies the Single Impact Condition, X' must occur outside of $B_R(0)$. Therefore, we must have $|X - X'| \geq \text{dist}(X, \partial B_R(0))$, which means that

$$\frac{u(X)}{\sqrt{u(X)^2 + |X - X'|^2}} \leq \frac{u(X)}{\sqrt{u(X)^2 + \text{dist}(X, \partial B_R(0))}}$$

Further, the differentiable function $z \to z/\sqrt{z^2 + \text{dist}(X, \partial B_R(0)}$ is strictly increasing since its derivative, $4/(z^2 + \text{dist}(X, \partial B_R(0))^{3/2}$, strictly positive for $z \geq 0$. Therefore, since $u(X) \leq H$, we can conclude that,

$$\frac{u(X)}{\sqrt{u(X)^2 + |X - X'|^2}} \leq \frac{H}{\sqrt{H^2 + \text{dist}(X, \partial B_R(0))}}$$

Substituting this into (3.3) gives

$$v_{f,3}(X) \geq |\overrightarrow{v_i}| \frac{-H}{\sqrt{H^2 + \text{dist}(X, \partial B_R(0))^2}}$$

And thus, by the order property of integration,

$$\begin{aligned} F(u) &= \int_{B_R(0)} \frac{1}{2|\overrightarrow{v_i}|} (1 + v_{f,3}(X))\, dX \\ &\geq \int B_R(0) \frac{1}{2} \left(1 - \frac{H}{H^2 + \text{dist}(X, \partial B_R(0))}\right) dX \\ &= \phi\left(B_R(0), H\right) \end{aligned}$$

As claimed. ■

3.3 Elementary Pairs: Definitions and Properties

3.3.1 Definitions and Symbols

In this subsection, we will define several symbols which will aid us in the subsequent sections. The symbols and definitions were borrowed from [Pla16].

Let A, B, C, and D be distinct points in $\mathbb{R}^2$ such that AB and CD are parallel. Suppose that $|A - B| > |C - D|$ and consider the trapezoid $ABCD$. Let $O \in \mathbb{R}^2$ be the point of intersection of the lines AC and BD. This is depicted in Figure 3.3.1.

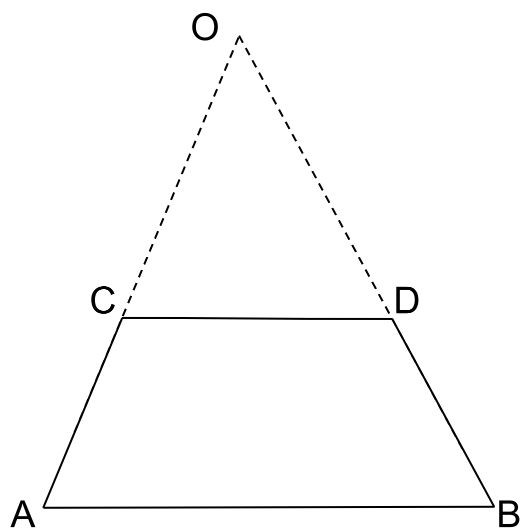

Figure 3.2: An elementary pair

Let $\square$ denote the (open) interior of the trapezoid $ABCD$. Let $\triangle$ denote the (open) interior of the triangle OCD.

Definition 3.1: The pair $(\square, \triangle)$ along with the point O is called an elementary pair.

Now, let d_M, d_m, and Γ be defined by

$$d_M = \sup\{|X - O| \mid X \in \square\} \tag{3.4}$$

$$d_m = \inf\{|X - O| \mid X \in \square\} \tag{3.5}$$

$$\Gamma = \frac{d_M - d_m}{d_M} \tag{3.6}$$

From these definitions and figure 3.3.1, we can deduce that

$$d_M = \max\{|O - A|, |O - B|\}$$

$$d_m = \text{dist}(O, \square) = \inf\{|O - X| : X \in \square\}$$

Now, let $p > 0$ be defined by,

$$p = \sqrt{d_M^2 + h^2} - h$$

Thus,

$$h = \frac{d_M^2 - p^2}{2p}$$

Finally, for each $X \in \square$, let $r(X)$ be the distance from X to O. That is,

$$r(X) = |X - O|$$

Notice that r has continuous partial derivatives for all $X \neq O$. Thus, r is differentiable on $\square$. Further, since $\square$ is open, we can conclude that for each $X \in \square$,

$$d_m < r(X) < d_M$$

Now, let $u_{H,\square} : \overline{\triangle \cup \square} \to \mathbb{R}$ be defined by

$$u_{H,\square}(X) = \begin{cases} \frac{r(X)^2 - p^2}{2p} & X \in \square \\ 0 & \text{otherwise} \end{cases}$$

Thus, $u_{H,\square}$ is zero on $\triangle$, as well as the boundary of $\square$ and $\triangle$. Since r is differentiable on $\square$, u must be differentiable on $\square$. u is also differentiable on $\triangle$ (since it is constant there). Thus, u is differentiable almost everywhere in $\overline{\triangle \cup \square}$.

3.3.2 Basic Properties of Elementary Pairs

In this subsection, we will prove some basic properties of elementary pairs. In particular, we will bound $u_{H,\square}$ and $1/(1 + |\nabla u_{H,\square}(X)|^2)$ on $\square$. These properties will help us prove that $\phi\,(B_R(0), H)$ is the infimum of $\{F(u) \mid u \in \mathfrak{S}_{B_R(0),H}\}$. As with most of this chapter, the theorems and arguments in this subsection were borrowed from [Pla16].

Theorem 3.2: For each $X \in \square$,

$$H - \Gamma\left(\sqrt{d_M^2 + H^2} + H\right) < u_{H,\square}(X) < H \tag{3.7}$$

Proof :

Let $X \in \square$. By definition,

$$d_m < r(X) < d_M$$

Therefore,

$$\frac{d_m - p^2}{2p} < \frac{r(X) - p^2}{2p} < \frac{d_M^2 - p^2}{2p} = H \tag{3.8}$$

Let's focus on the left side of this inequality. Notice that,

$$\begin{aligned}\frac{d_m - p^2}{2p} &= \frac{d_M^2 - p^2}{2p} - \frac{d_M^2 - d_m^2}{2p} \\ &= H - \frac{(d_M + d_m)(d_M - d_m)}{2p}\end{aligned}$$

However, by definition, $p = \sqrt{d_M^2 + H^2} - H$. Further, notice that

$$d_M - d_m = d_M(d_M - d_m)/d_M = d_M\Gamma$$

Finally, since $d_m \leq d_M$, we must have $(d_m + d_M)/2 \leq d_M$. Thus,

$$\frac{d_m - p^2}{2p} \geq H - \Gamma\left(\frac{d_M^2}{\sqrt{d_M^2 + H^2} - H}\right)$$

Moreover,

$$\begin{aligned}\frac{d_M^2}{\sqrt{d_M^2 + H^2} - H} &= \frac{d_M^2\left(\sqrt{d_M^2 + H^2} + H\right)}{d_M^2} \\ &= \sqrt{d_M^2 + H^2} + H\end{aligned}$$

And thus,

$$\frac{d_m - p^2}{2p} \geq H - \Gamma\left(\sqrt{d_M^2 + H^2} + H\right)$$

Substituting this back into (3.8) gives

$$H - \Gamma\left(\sqrt{d_M^2 + H^2} + H\right) < u_{H,\square}(X) < H$$

Since we chose X arbitrarily, this must hold for each $X \in \square$. ■

Equation (3.7) tells us that $u_{H,\square}$ is bounded above by H. The lower bound of this equation, therefore, gives us a useful criterion for whether $u_{H,\square} \in \mathfrak{S}_{B_R(0),H}$. In particular, we can see that if

$$\Gamma \leq \frac{H}{H + \sqrt{H^2 + d_M^2}} \tag{3.9}$$

then $H - \Gamma\left(\sqrt{d_M^2 + H^2} + H\right) \geq 0$. Thus, if Γ satisfies (3.9), then $u_{H,\square}$ must be non-negative on $\square$, which means that it is non-negative on $\overline{\square \cup \triangle}$ and is, therefore, an element of $\mathfrak{S}_{B_R(0),H}$. Henceforth, we shall refer to (3.9) as the "non-negativity criterion". With that established, let us now turn our attention to $1/(1 + |\nabla u_{H,\square}(X)|^2)$.

Theorem 3.3: For each $X \in \square$,

$$\frac{1}{2}\left(1 - \frac{d_M}{\sqrt{d_M^2 + H^2}}\right) < \frac{1}{1 + |\nabla u_{H,\square}(X)|^2} < \frac{(1-\Gamma)^{-2}}{2}\left(1 - \frac{H}{\sqrt{d_M^2 + H^2}}\right) \tag{3.10}$$

Proof :

To begin, let $X = (x, y) \in \square$. Further, let $O = (O_x, O_y)$. Then, by definition,

$$r(X) = r(x, y) = |(x - O_x, y - O_y)|$$

and

$$r(X)^2 = (x - O_x)^2 + (y - O_y)^2$$

Further, notice that

$$\begin{aligned}
|\nabla u_{H,\square}(X)| &= \left|\left(\mathrm{D}_x\, u_{H,\square}(X), \mathrm{D}_y\, u_{H,\square}(X)\right)\right| \\
&= \left|\frac{1}{2p}\left(2(x - O_x), 2(y - O_y)\right)\right| \\
&= \frac{r(X)}{p}
\end{aligned}$$

Since $d_m < r(X) < d_M$, we must have

$$\frac{p^2}{d_M^2 + p^2} < \frac{1}{1 + \nabla u_{H,\square}(X)|^2} < \frac{p^2}{d_m^2 + p^2} \tag{3.11}$$

Let's focus on the left side of (3.11). Notice that,

$$\frac{p^2}{d_M^2 + p^2} = \frac{p}{2p + \frac{d_M^2 - p^2}{p}}$$

However, we know that $H = (d_M^2 - p^2)/(2p)$. Thus, $\frac{d_M^2 - p^2}{p} = 2H$, which means that

$$\frac{p^2}{d_M^2 + p^2} = \frac{1}{2}\left(\frac{p}{p + h}\right)$$

And, by definition, $p = \sqrt{d_M^2 + H^2} - H$. Thus,

$$\frac{p^2}{d_M^2 + p^2} = \frac{1}{2}\left(1 + \frac{H}{\sqrt{d_M^2 + H^2}}\right) \tag{3.12}$$

Now let's focus on the right side of (3.11). To begin, notice that

$$d_m = \frac{d_m}{d_M} d_M = \frac{d_m - d_M + d_M}{d_M} d_M = \frac{1 - \Gamma}{d_M} d_M$$

Thus,

$$\frac{p^2}{d_m^2 + p^2} = \frac{p^2}{\left(\frac{1-\Gamma}{d_M} d_M\right)^2 + p^2}$$
$$= \frac{p^2 (1-\Gamma)^{-2}}{d_M^2 + \frac{p^2}{(1-\Gamma)^2}}$$

However, notice that $1 - \Gamma = d_m/d_M$, which must be an element of $(0, 1]$. Thus,

$$\frac{p^2}{(1-\Gamma)^2} \leq p^2$$

which means that

$$\frac{p^2}{d_m^2 + p^2} \leq \frac{p^2 (1-\Gamma)^{-2}}{d_M^2 + p^2}$$

And thus, by (3.12),

$$\frac{p^2}{d_m^2 + p^2} \leq \frac{(1-\Gamma)^{-2}}{2} \left(1 - \frac{H}{d_M^2 + H^2}\right) \tag{3.13}$$

Substituting equations (3.12) and (3.13) into (3.11) gives the desired result. ■

Importantly, if we make Γ sufficiently small, then we can make the upper and lower bounds in (3.10) very close to one another. In other words, for small Γ, we can place tight bounds on $F(u)$. The bounds in (3.10) should look familiar. They closely resemble the integrand of $\phi\left(B_R(0), H\right)$ in (3.2), but with d_M in the place of $\operatorname{dist}(X, \partial B_R(0))$. Further, the upper bound has an extra factor of $(1-\Gamma)^{-2}$. Thus, if we can make d_M sufficiently close to $\operatorname{dist}(X, \partial B_R(0))$, and Γ sufficiently close to 0, then we can make $F(u)$ very close to $\phi\left(B_R(0), H\right)$. This, in essence, is the approach that we will use in the following sections and the one that Plakhov, [Pla19], first used to prove $\phi\left(B_R(0), H\right) = \inf\{F(u) \mid u \in \mathfrak{S}_{B_R(0),H}\}$.

3.3.3 Revisiting the Single Impact Condition

In this subsection, we will show show that the body $B_{u_{H,\square}}$ satisfies the Single Impact Condition. We will then show, in essence, that combining a finite number of $u_{H,\square_i}$ yields a body that also satisfies the Single Impact Condition.

Theorem 3.4: If $u_{H,\square}$ is non-negative, then it satisfies the Single Impact Condition.

Proof :

Geometrically, this works because the part of $u_{H,\square}$ over $\square$ is a part of a paraboloid whose focus is O. Thus, particles that reflect off of this part of $B_{u_{H,\square}}$ pass through point O without interacting with the rest of the body. The rest of $B_{u_{H,\square}}$ is flat, which means that incoming particles are reflected away vertically.

Let us now make this geometric argument more rigorous. For brevity, we will place the origin of the (x, y, z) coordinate system at O. Suppose that $(x, y) = X \in \square$. Then, $r(X) = |X|$, which means that

$$u_{H,\square}(X) = \frac{1}{2p}\left(|X|^2 - p^2\right)$$

Further, in the proof of Theorem 3.3, we showed that $\nabla u_{H,\square}(X) = \frac{1}{p}\left(x - o_x, y - o_y\right)$, where o_x and o_y are the x and y components of O, respectively. Since $O = (0, 0)$, we must have

$$\nabla u_{H,\square}(X) = \frac{X}{p}$$

And thus, the Single Impact Condition (1.27) becomes,

$$\frac{u\left(X - t\frac{X}{p}\right) - u_{H,\square}(X)}{t} \leq \frac{1}{2}\left(1 - \frac{|X|^2}{p^2}\right) \tag{3.14}$$

for all $t > 0$ such that $X - t\left(\frac{X}{p}\right) \in \overline{\square \cup \triangle}$. Notice that when $t = p$, $X - t\left(\frac{X}{p}\right) = 0 = O$, which is in $\overline{\square \cup \triangle}$. Since this set is convex, the line between O and X must be in $\overline{\square \cup \triangle}$. Thus, in our case, $0 < t \leq p$. With some rearranging, (3.14) becomes,

$$u\left(\left(\frac{p-t}{p}\right)X\right) - u_{H,\square}(X) = \frac{t}{2}\left(1 - \frac{|X|^2}{p^2}\right) \tag{3.15}$$

For $0 < t \leq p$. Now let $t \in (0, p]$. In general, either $((p-t)/p)X \in \overline{\triangle}$ or not. If so, then $u\left(\left(\frac{p-t}{p}\right)X\right) = 0$, and (3.15) becomes

$$-u_{H,\square}(X) \leq \frac{t}{2}\left(1 - \frac{|X|^2}{p^2}\right) \tag{3.16}$$

However, notice that

$$\begin{aligned}
\frac{t}{2}\left(1 - \frac{|X|^2}{p^2}\right) &= \frac{t}{p}\left(\frac{p}{2} - \frac{|X|^2}{2p}\right) \\
&= -\frac{t}{p}\left(\frac{|X|^2 - p^2}{2p}\right) \\
&= -\left(\frac{t}{p}\right) u_{H,\square}(X)
\end{aligned}$$

And thus, (3.16) becomes $-u_{H,\square}(X) \leq -(t/p)u_{H,\square}(X)$, which is obviously true because $u_{H,\square}$ is non-negative, and $0 < t \leq p$. Now suppose that $X \notin \overline{\triangle}$, which means that $X \in \square$. Thus,

$$\begin{aligned}
u\left(\left(\frac{p-t}{p}\right)X\right) &= \frac{\left|\left(\frac{p-t}{p}\right)X\right|^2 - p^2}{2p} \\
&= \frac{1}{2p}\left(\frac{(t-p)^2}{p^2}|X|^2 - p^2\right)
\end{aligned}$$

And thus,

$$u\left(\left(\frac{p-t}{p}\right)X\right) - u_{H,\square}(X) = \frac{1}{2p}\left(\frac{(t-p)^2}{p^2}|X|^2 - |X|^2\right)$$

We will show that, in this case, a simple inequality implies (3.15). Since $t \leq p$, we must have

$$(t-p)\frac{|X|^2}{2p^3} \leq 0 \leq \frac{1}{2}$$

Multiplying this by $t > 0$ gives

$$(t^2 - tp)\frac{|X|^2}{2p^3} \leq \frac{t}{2}$$

However, $t^2 - pt = (t-p)^2 - p^2 + tp$. Thus, we must have

$$\frac{1}{2p^3}|X|^2\left((t-p)^2 - p^2\right)t^2 \leq \frac{t}{2}\left(1 - \frac{|X|^2}{p^2}\right)$$

But,

$$\frac{1}{2p^3}|X|^2\left((t-p)^2 - p^2\right) = \frac{1}{2p}\left(\frac{(t-p)^2}{p^2}|X|^2 - |X|^2\right)$$

Therefore,

$$u\left(\left(\frac{p-t}{p}\right)X\right) \leq \frac{t}{2}\left(1 - \frac{|X|^2}{p^2}\right)$$

Thus in either case, (3.15) holds. Since we chose t arbitrarily, we can conclude that u satisfies the Single Impact Condition if $X \in \square$. If, by contrast, $X \in \triangle$, then $B_{u_{H,\square}}$ is flat around x. Thus, the particle reflects vertically off of $B_{u_{H,\square}}$, and will not collide with the body again. In particular, $\nabla u_{H,\square}(X) = 0$, which means that for all $t > 0$,

$$0 = u_{H,\square}(X) - u_{H,\square}(X) < \frac{t}{2} = \frac{t}{2}\left(1 - |\nabla u_{H,\square}(X)|^2\right)$$

Thus, the particle satisfies the Single Impact Condition. Therefore, $u_{H,\square}$ satisfies the Single Impact Condition. * ■

*We can ignore the cases when X is on the boundary of $\triangle$ or $\square$. The boundary has measure zero, which means that the probability that an individual particle collides with the border is 0.

Let $n \in \mathbb{N}$ and consider a collection of n elementary pairs, $\{(\Box_i, \triangle_i)\}_{i=1}^n$ with foci $\{O_i\}_{i=1}^n$. For each $i \in \{1, 2...n\}$, let $u_i = u_{H,\Box_i} : \overline{\Box_i \cup \triangle_i} \to \mathbb{R}$ be non-negative. Suppose that $B_R(0) \subset \cup_{i=1}^n \overline{\Box_i \cup \triangle_i}$ and that each $O_i \notin B_R(0)$. Let $u : \overline{B_R(0)} \to \mathbb{R}$ be defined by

$$u(X) = \min\{u_i(X) \mid X \in \overline{\Box_i \cup \triangle_i}\} \tag{3.17}$$

Theorem 3.5: u satisfies the Single Impact Condition.

Proof :

Suppose that a particle hits the body B_u, and let $X \in B_R(0)$ denote the (x, y) coordinates of the point of impact. Either X is in one of the triangles or not. Suppose that X is in one of the triangles. Since each u_i is non-negative, and since each u_i is zero in its corresponding triangles, we must have $u(X) = 0$. Since the triangles are open, X must be in the interior of a triangle. Thus, $u = 0$ on some ball around X, which means that $\nabla u(X) = 0$. Therefore, the particle reflects away vertically and, therefore, satisfies the Single Impact Condition. Now suppose that X is not in any of the triangles. Then X must be in the interior of at least one trapezoid. Let i denote the index of the trapezoid which corresponds to $\min\{u_i(X) \mid X \in \overline{\Box_i \cup \triangle_i}\}$. Then, $u(X) = u_i(X)$. Arguing as we did in the proof of Theorem 3.4, we can conclude that the particle passes through the xy plane at O_i. By Theorem 3.4, we know that the particle will remain above B_{u_i} as it travels from X to O_i through the ith elementary pair. Further, because of how we defined u, $u \leq u_i$ on the ith elementary pair. In particular, for each $t \in (0, p]$,

$$u\left(X - t\nabla u(X)\right) \leq u_i\left(X - t\nabla u(X)\right)$$

In other words, u is no taller than u_i along the particle's trajectory, which means that

$$\begin{aligned} u\left(X - t\nabla u(X)\right) - u(X) &\leq u_i\left(X - t\nabla u(X)\right) - u(X) \\ &= u_i\left(X - t\nabla u_i(X)\right) - u_i(X) \\ &\leq \frac{t}{2}\left(1 - |\nabla u(X)|^2\right) \end{aligned}$$

And thus, the particle satisfies the Single Impact Condition. Therefore, in either case, the particle collision satisfies the Single Impact Condition. Since we chose this collision arbitrarily, we can conclude that u satisfies the Single Impact Condition. ■

3.4 Almost Optimal Solutions

In this section, we will show that the results that we proved in the previous section, along with a critical result in [Pla16], leads us to the conclusion $\phi\left(B_R(0), H\right) = \inf\{F(v) : v \in \mathfrak{S}_{B_R(0),H}\}$.

Plakhov, [Pla16], proved the following theorem (which is Lemma 4 in [Pla16]), †

Theorem 3.6: For any $\epsilon > 0$, there exists a finite family of elementary pairs $\{(\square_i, \triangle_i)\}_{i=1}^n$ with ratios Γ_i and foci O_i such that

1. $B_R(0) \subset \cup_{i=1}^n \overline{\square_i \cup \triangle_i}$
2. $|\cup_{i=1}^n \triangle_i| < \epsilon$

†(6) and (7) in Theorem 3.6 are not a part of the statement of Lemma 4 in Plakhov. However, Plakhov uses his Lemma 4 as if they were a part of it. These results purportedly follow from Plakhov's proof of Lemma 5. Therefore, for completeness, I included both conditions in the statement of Theorem 3.6.

3. For each $i \in \{1, 2...n\}$, $\Gamma_i < \epsilon$

4. For each $i \in \{1, 2...n\}$, $O_i \notin B_R(0)$

5. For each $i \in \{1, 2...n\}$ and $X \in \Box_i$, $|X_i - O_i| < \text{dist}(X, \partial B_R(0)) + \epsilon$

6. For each $i \in \{1, 2...n\}$, $d_{M,i} \leq R + \epsilon$

7. For each $i \in \{1, 2...n\}$ and $X \in \Box_i$, $d_{M,i} \leq d(X, \partial B_R(0)) + \epsilon$

The proof of Theorem 3.6 is rather involved and uses the same approach that Besicovitch used to solve the Kakeya problem. The Kakeya problem is to find the smallest subset of $\mathbb{R}^2$ through which a needle of unit length can be turned around. The interested reader can find out more about the Kakeya problem and Besicovitch technique in [Bes63] or [Cun71].

Before we move onto Theorem 3.7, it is worth considering what Theorem 3.6 tells us. The (2) in Theorem 3.6 essentially says that the trapezoids are very narrow compared with the length of its corresponding elementary pair. In other words, the trapezoids only occupy the bottom sliver of each elementary pair. The (3) and (4) in Theorem 3.6 collectively tell us that each focus is outside of $B_R(0)$, but is very close to the border of $B_R(0)$. With this in mind, we can now prove the final theorem of this thesis,

Theorem 3.7: For sufficiently small $\epsilon > 0$, there is some $u_\epsilon \in \mathfrak{S}_{B_R(0),H}$ and some $V \subset B_R(0)$ such that

- $|V| < \epsilon$
- For each $i \in \{1, 2...n\}$, $\text{dist}(X, \partial B_R(0)) < \epsilon$

- $$F(u_\epsilon) < |V| + \frac{(1+\epsilon)^{-2}}{2}\int_{B_R(0)-V}\left(1 - \frac{H}{\sqrt{H^2 + (\mathrm{dist}(X, \partial B_R(0)) + \epsilon)^2}}\right) dX \quad (3.18)$$

Proof :

To begin, since $R > 0$ (R is the radius of the ball $B_R(0)$), we must have

$$0 < \frac{1}{1 + \sqrt{1 + \frac{(R+1)^2}{H^2}}}$$

Now, pick $\epsilon > 0$ such that

$$\epsilon < \min\left\{1, \frac{1}{1 + \sqrt{1 + \frac{(R+1)^2}{H^2}}}\right\}$$

Let $(\square_i, \triangle_i)_{i=1}^n$ be the collection of elementary pairs guaranteed by Theorem 3.6. On each elementary pair, let $u_i = u_{H,\square_i}$ and let $u_\epsilon \overline{B_R(0)} \to \mathbb{R}$ be as defined in (3.17). Since $\epsilon < 1$, (6) in Theorem 3.6 tells us that for each $i \in \{1, 2...n\}$,

$$d_{M,i} \leq R + \epsilon < R + 1$$

And thus,

$$\epsilon < \frac{1}{1 + \sqrt{1 + \frac{(R+1)^2}{H^2}}} < \frac{1}{1 + \sqrt{1 + \frac{d_{M,i}^2}{H^2}}}$$

This implies that each elementary pair satisfies the non-negativity criterion (3.9). In particular, this means that u_ϵ satisfies the non-negativity criterion. Therefore, by Theorem 3.5, $u_\epsilon \in \mathfrak{S}_{B_R(0),H}$. Now, let $i \in \{1, 2, ..n\}$ and suppose that that $X \in B_R(0)$. Consider the line segment $[X, O_i]$ Since $X \in B_R(0)$ and $O_i \notin B_R(0)$, there must be some point $Y \in [X, O_i]$ such that $Y \in \partial B_R(0)$. Importantly,

$$|X - O_i| = |X - Y| + |Y - O_i|$$

By definition, $\operatorname{dist}(X, \partial B_R(0)) = \inf\{|X - Z| : Z \in \partial B_R(0)\}$. Thus,

$$|X - Y| \geq \operatorname{dist}(X, \partial B_R(0))$$

Further, by (5) in Theorem 3.6,

$$|X - O_i| < \operatorname{dist}(X, \partial B_R(0)) + \epsilon$$

Therefore,

$$\operatorname{dist}(O_i, B_R(0)) \leq |O_i - Y| = |X - O_i| - |X - Y| < \epsilon$$

Since we chose i arbitrarily, this must hold for each $i \in \{1, 2 ... n\}$. Now, let

$$V = (\cup_{i=1}^{n} \triangle_i) \cap B_R(0)$$

Thus, $|V| \leq |\cup_{i=1}^{n} \triangle_i| < \epsilon$. If $X \in V$, then we must have $\nabla u_\epsilon(X) = 0$, which means that

$$\frac{1}{1 + |\nabla u_\epsilon|^2} = 1$$

If $X \in B_R(0) - V$, then X must be contained in at least one of the trapezoids. Let i denote the index of the trapezoid which corresponds to $\min\{u_i(X) \mid X \in \overline{\square_i \cup \triangle_i}\}$. Then, $\nabla u_\epsilon(X) = \nabla u_i(X)$, which means that

$$\frac{1}{1 + |\nabla u_\epsilon(X)|^2} = \frac{1}{1 + |\nabla u_i(X)|^2}$$

However, by Theorem 3.3,

$$\frac{1}{1 + |\nabla u_i(X)|^2} < \frac{(1 - \Gamma_i)^{-2}}{2} \left(1 - \frac{H}{\sqrt{d_{M,i}^2 + H^2}} \right)$$

Further, by (3) in Theorem 3.6, we know that $\Gamma_i < \epsilon$. Thus,

$$\frac{(1-\Gamma_i)^{-2}}{2} < \frac{(1-\epsilon)^{-2}}{2}$$

Further, by (7) in Theorem 3.6, we know that

$$d_{M,i} \leq d(X, \partial B_R(0)) + \epsilon$$

which means that

$$1 - \frac{H}{\sqrt{H^2 + d_{M,i}^2}} \leq 1 - \frac{H}{\sqrt{H^2 + \left(\operatorname{dist}(X, \partial B_R(0)) + \epsilon\right)^2}}$$

Therefore, we must have

$$\frac{1}{1+|\nabla u_\epsilon(X)|^2} < \frac{(1-\epsilon)^{-2}}{2}\left(1 - \frac{H}{\sqrt{H^2 + \left(\operatorname{dist}(X, \partial B_R(0)) + \epsilon\right)^2}}\right)$$

Since we chose X arbitrarily, this must hold for each $X \in B_R(0) - V$. Therefore, using the properties of integration

$$\begin{aligned}
F(u_\epsilon) &= \int_{B_R(0)} \frac{1}{1+|\nabla u_\epsilon(X)|^2} dX \\
&= \int_V \frac{1}{1+|\nabla u_\epsilon(X)|^2} dX + \int_{B_R(0)-V} \frac{1}{1+|\nabla u_\epsilon(X)|^2} dX \\
&< \int_V 1 dX + \int_{B_R(0)-V} \frac{(1-\epsilon)^{-2}}{2}\left(1 - \frac{H}{\sqrt{H^2 + \left(\operatorname{dist}(X, \partial B_R(0)) + \epsilon\right)^2}}\right) dX \\
&= |V| + \frac{(1-\epsilon)^{-2}}{2} \int_{B_R(0)-V} \left(1 - \frac{H}{\sqrt{H^2 + \left(\operatorname{dist}(X, \partial B_R(0)) + \epsilon\right)^2}}\right) dX
\end{aligned}$$

As claimed. ■

Importantly, we know that V has measure $< \epsilon$. Further, as $\epsilon \to 0$,

$$\frac{H}{\sqrt{H^2 + (\mathrm{dist}(X, \partial B_R(0) + \epsilon)^2}} = \frac{H}{\sqrt{H^2 + \mathrm{dist}(X, \partial B_R(0))^2}} + O(\epsilon)$$

Similarly, as $\epsilon \to 0$,

$$\frac{(1-\epsilon)^2}{2} = \frac{1}{2} + O(\epsilon)$$

Therefore,

$$\begin{aligned} F(u_\epsilon) &< |V| + \frac{(1-\epsilon)^2}{2} \int_{B_R(0)-V} \left(1 - \frac{H}{\sqrt{H^2 + (\mathrm{dist}(X, \partial B_R(0)) + \epsilon)^2}}\right) dX \\ &= \epsilon + \left(\frac{1}{2} + O(\epsilon)\right) \int_{B_R(0)-V} \left(1 - \frac{H}{\sqrt{H^2 + \mathrm{dist}(X, \partial B_R(0))^2}} + O(\epsilon)\right) dX \end{aligned}$$

And thus,

$$F(u_\epsilon) = \phi\left(B_R(0), H\right) + O(\epsilon)$$

Therefore, we can conclude that

$$\phi\left(B_R(0), H\right) = \inf\{F(u) \mid u \in \mathfrak{S}_{B_R(0),H}\}$$

Figure 3.3, which was borrowed from [Pla16], roughly depicts B_{u_ϵ}.

3.5 Final Remarks

In the previous section, we showed that $\phi\left(B_R(0), H\right) = \inf\{F(u) \mid u \in \mathfrak{S}_{B_R(0),H}\}$. We also found some functions whose resistance is very close to this infimum. However, the bodies that we used in the proof of Theorem 3.7 are not practical. As discussed in [Pla16], these bodies are incredibly complicated when ϵ is small. For them to have a lower resistance

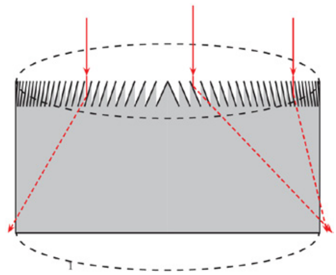

Figure 3.3: An almost optimal body.

than Newton's rotationally symmetric body when $R = 1$m, they must have surface features that are smaller than an atom. As such, it is not possible to manufacture these bodies in the real world. Therefore, the proof of Theorem 3.7 does not tell us how to design low resistance bodies; it merely proves the existence of a result. In theory, there may be other, more practical bodies whose resistance is arbitrarily close to $\phi\left(B_R(0), H\right)$.

Further, it is worth noting that the fluid model that we used to derive F does not apply to the bodies that we used to prove Theorem 3.6 and 3.7. Both Newton's fluid model and the elastic fluid model assume that fluid particles do not interact. However, in the proof of Theorem 3.5, we saw that bodies that we used to prove Theorem 3.6 channel most of their particles through a finite number of foci. Since there are many particles, the fluid particles undoubtedly interact near the foci. Therefore, the fluid model that we used to derive the Minimum Resistance Problem does not apply to the bodies that we used to prove Theorem 3.7. There may be bodies in $\mathfrak{S}_{B_R(0),H}$ whose resistance is very close to $\phi\left(B_R(0), H\right)$ that do not cause fluid particles to interact. However, at the time of writing this thesis, whether or not such bodies exist is unknown.

The fact that $\mathfrak{S}_{B_R(0),H}$ contains bodies that violate the fluid model underlying the

Minimum Resistance Problem suggests that we should formulate a criterion that guarantees particles do not interact. We could then consider the Minimum Resistance Problem on the set of bodies that satisfy that criterion (like we did for the Single Impact Condition). As far as I am aware, however, no such criterion is known at the time of writing this thesis. The search for and analysis of such a criterion, therefore, represents a potential area of future research.

Because of these shortcomings, I believe that this variant of the Minimum Resistance Problem is far from being completely understood. It is impressive that arguably the oldest problem in the Calculus of Variations continues, to this day, to be an active area of research. I hope that this thesis gives you an idea of the impressive body of work that has come from Newton's innocuous proposition in *Philosophiæ Naturalis Principia Mathematica* on the design of ships.

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